

Class 12

The Risk-Neutral Measure

A stock that pays no dividends over $[0, T]$ and a money-market account follow

$$\frac{dS}{S} = \mu dt + \sigma dB, \quad \frac{d\beta}{\beta} = r dt, \quad \frac{d\Lambda}{\Lambda} = -r dt - \lambda dB,$$

where only the part of the SDF Λ correlated with the stock matters for pricing it. The pricing equation makes $\lambda = (\mu - r)/\sigma$ the stock's Sharpe ratio. The process $\mathcal{E} = \Lambda\beta$ satisfies $d\mathcal{E}/\mathcal{E} = -\lambda dB$, so it is a positive (local) martingale, and when it is a true martingale it defines the **risk-neutral measure** $dP^*/dP = \mathcal{E}_T/\mathcal{E}_0$. By Girsanov's theorem, $dB^* = dB + \lambda dt$ is a P^* -Brownian motion, and

$$\frac{dS}{S} = r dt + \sigma dB^*. \quad (1)$$

Every asset V driven by the same B has the same Sharpe ratio, so $dV/V = r dt + \sigma_V dB^*$: under P^* all assets earn the risk-free rate. Since ΛV is a martingale,

$$V_0 = E\left(\frac{\Lambda_T}{\Lambda_0} V_T\right) = E^*\left(e^{-\int_0^T r_s ds} V_T\right). \quad (2)$$

Prices are expected payoffs discounted at the risk-free rate, provided the expectation uses P^* .

Bonds, Futures, and Forwards

A zero-coupon bond has $V_T = 1$, so $P(T) = E^*(e^{-\int_0^T r_s ds})$. A futures position costs nothing to enter and pays its gains dF continuously, so $E(\Lambda dF) = 0$, or $E^*(dF) = 0$: the futures price is a P^* -martingale, and $F(T) = E^*(S_T)$. The forward price $\varphi(T)$ gives the forward contract zero value, which yields

$$\varphi(T) = F(T) + \frac{\text{Cov}^*\left(e^{-\int_0^T r_s ds}, S_T\right)}{P(T)}. \quad (3)$$

Forward and futures prices coincide when this covariance is zero, in particular when r is constant.

The Black-Scholes Formula

With constant parameters, (1) gives $S_T = S_0 e^{(r - \frac{1}{2}\sigma^2)T + \sigma B_T^*}$, so $\ln S_T$ is normal under P^* with mean $\ln S_0 + (r - \frac{1}{2}\sigma^2)T$ and variance $\sigma^2 T$. Splitting the call payoff into $S_T \mathbf{1}_{\{S_T > K\}} - K \mathbf{1}_{\{S_T > K\}}$,

$$C_0 = S_0 P^S(S_T > K) - K e^{-rT} P^*(S_T > K), \quad (4)$$

where P^S is the measure with density $\Lambda_T S_T / (\Lambda_0 S_0)$, which uses the stock as the unit of account (a **change of numeraire**). Under P^S the stock has expected return $r + \sigma^2$, so only the drift differs across the three measures, as Figure 1 shows. Evaluating both probabilities,

$$C = S_0 N(d_1) - K e^{-rT} N(d_2), \quad P = K e^{-rT} N(-d_2) - S_0 N(-d_1), \quad (5)$$

with

$$d_1 = \frac{\ln(S_0/K) + \left(r + \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}, \quad d_2 = d_1 - \sigma\sqrt{T},$$

and $N(d)$ the standard normal CDF. Here $N(d_2)$ is the risk-neutral probability that the call is exercised. The expected return μ never appears. Subtracting the two prices gives put-call parity, $C - P = S_0 - K e^{-rT}$, which holds in any model.

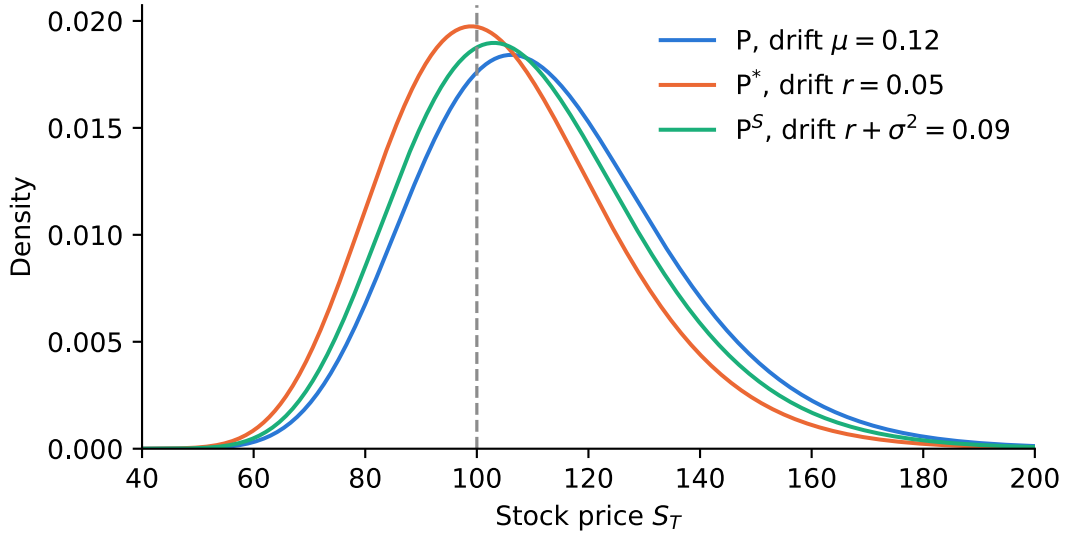


Figure 1: Density of the stock price S_T after one year under the three measures, with $S_0 = 100$, $\sigma = 0.20$, $\mu = 0.12$, and $r = 0.05$. The volatility is the same under all three; only the drift changes. The dashed line marks a strike of 100.

The Black-Scholes PDE

If $V(S, t)$ is the price of a derivative paying $f(S_T)$, Itô's lemma under (1) gives its drift, and that drift must equal rV :

$$rS \frac{\partial V}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t} = rV, \quad V(S, T) = f(S). \quad (6)$$

All derivatives on S solve the same equation and differ only in the terminal condition. The diffusion term $\sigma S \partial V / \partial S$ shows that holding the derivative and selling $\partial V / \partial S$ shares (its **delta**) removes all exposure to dB , which was the original argument of Black and Scholes. The solution of (6) is the expectation (2), the Feynman-Kac formula.

The Merton Model

Real prices gap. Let N be a Poisson process with intensity λ (from here on, λ is a jump intensity) and Y_i iid $\mathcal{N}(m, s^2)$ log jump sizes, independent of B . In the model of Merton,

$$S_t = S_0 \exp \left(\left(\mu - \frac{1}{2} \sigma^2 - \lambda k \right) t + \sigma B_t + \sum_{i=1}^{N_t} Y_i \right), \quad k = E(e^Y) - 1 = e^{m + \frac{1}{2} s^2} - 1. \quad (7)$$

Each jump multiplies the price by e^{Y_i} , and k is the expected percentage jump. The $-\lambda k$ correction keeps $E(S_t) = S_0 e^{\mu t}$. Given $N_t = n$, the log return is normal, so its density is a Poisson mixture of normals. Every cumulant beyond the variance comes from the jumps:

$$\kappa_2 = \sigma^2 t + \lambda t(m^2 + s^2), \quad \kappa_3 = \lambda t(m^3 + 3ms^2), \quad \kappa_4 = \lambda t(m^4 + 6m^2 s^2 + 3s^4). \quad (8)$$

With $m < 0$ the skewness $\kappa_3/\kappa_2^{3/2}$ is negative and the excess kurtosis κ_4/κ_2^2 is positive. They shrink like $1/\sqrt{t}$ and $1/t$, so jumps matter most over short horizons.

Itô's Formula with Jumps and Changes of Intensity

Compensated jumps. If the Z_i are iid and independent of N , then $\sum_{i \leq N_t} Z_i - \lambda E(Z)t$ is a martingale. A process with jumps is a martingale when its continuous drift plus λ times its expected jump is zero.

Itô's formula. For $X_t = X_0 + \int a_s ds + \int b_s dB_s + \sum_{i \leq N_t} Z_i$ and f twice continuously differentiable,

$$df(X_t) = \left(af'(X_t) + \frac{1}{2} b^2 f''(X_t) \right) dt + bf'(X_t) dB_t + (f(X_{t-} + \Delta X_t) - f(X_{t-})). \quad (9)$$

The jump term is the exact change in f , not a Taylor expansion. Applied to (7), $dS_t/S_{t-} = (\mu - \lambda k) dt + \sigma dB_t + (e^Y - 1) dN_t$, whose jump term has compensator $\lambda k dt$.

Changing the intensity. For $\lambda^* > 0$, the positive martingale

$$Z_t = \left(\frac{\lambda^*}{\lambda} \right)^{N_t} e^{-(\lambda^* - \lambda)t} \quad (10)$$

defines a measure under which N is Poisson with intensity λ^* . Brownian reweighting only shifts a drift, but the quadratic variation of N is random, so reweighting paths can change how often jumps occur. Jump sizes can be reweighted too.

Credit Spreads

A firm defaults at the first event of N , and bondholders lose a fraction L at default. If the SDF jumps by the factor $1 + \eta$ at default, $\Lambda_t = e^{-rt}(1 + \eta)^{N_t} e^{-\lambda \eta t}$, then $\Lambda \beta$ is the density (10) with $\lambda^* = \lambda(1 + \eta)$, and a bond priced $e^{-c(T-t)}$ before default has spread

$$c - r = \lambda^* L = (1 + \eta) \lambda L. \quad (11)$$

With $\eta > 0$, defaults hit in bad times, and the spread exceeds the expected loss rate λL .

Option Pricing with Jumps

A stock position hedges dB but not a jump of random size, because an option's value changes nonlinearly in the jump. The market is **incomplete**: many risk-neutral measures price the stock, each with $dS_t/S_{t-} = (r - \lambda^* k^*) dt + \sigma dB_t^* + (e^Y - 1) dN_t$. Merton assumed that jump risk is diversifiable and unpriced, so λ and the jump distribution are the same under P and P^* . The call price is then a Poisson mixture of Black-Scholes prices,

$$C = \sum_{n=0}^{\infty} e^{-\lambda' T} \frac{(\lambda' T)^n}{n!} C^{BS}(S_0, K, r_n, \sigma_n, T), \quad (12)$$

with $\lambda' = \lambda(1 + k)$, $\sigma_n^2 = \sigma^2 + n s^2/T$, and $r_n = r - \lambda k + n \ln(1 + k)/T$. Downward jumps raise the implied volatility of low strikes, producing the **skew** of Figure 2, which is steepest at short maturities.

For models without a formula like (12), (4) still holds, and both probabilities come from the risk-neutral characteristic function ϕ of $\ln S_T$ by the inversion formula $P(X > x) = \frac{1}{2} + \frac{1}{\pi} \int_0^{\infty} \text{Im}(e^{-iux} \phi(u))/u du$, using $\phi^S(u) = \phi(u - i)/\phi(-i)$ for P^S .

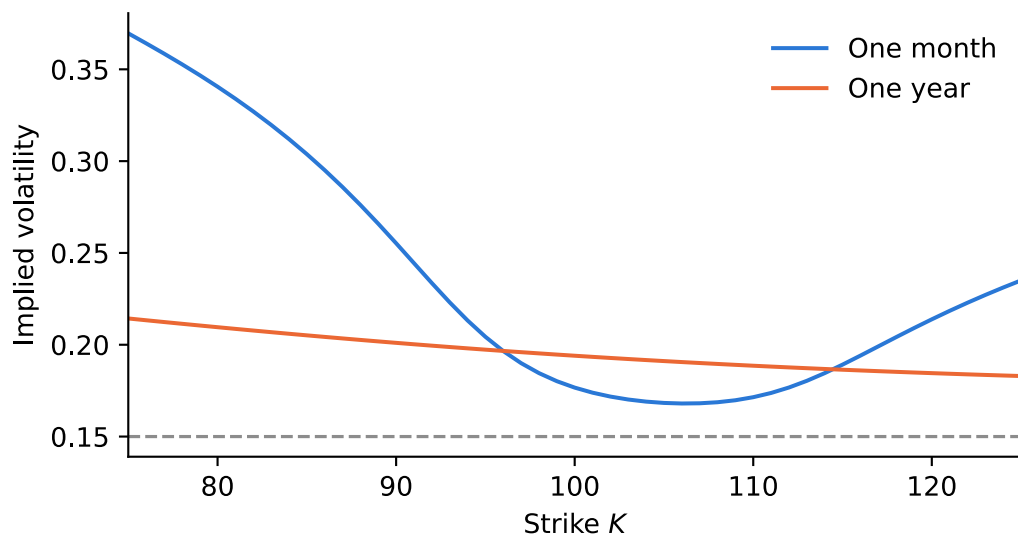


Figure 2: Black-Scholes implied volatility of Merton call prices against the strike for one-month and one-year options, with $S_0 = 100$, $r = 0.05$, $\sigma = 0.15$, $\lambda = 1$, $m = -0.08$, and $s = 0.10$. The dashed line is the diffusion volatility.

Practice Problems

1. A stock follows $dS/S = \mu dt + \sigma dB$ with $\mu = 0.10$ and $\sigma = 0.20$, its price today is $S_0 = 100$, and the risk-free rate is a constant $r = 0.04$.
 - a. Compute λ and write the dynamics of S under P^* .
 - b. Another asset is driven by the same Brownian motion and has volatility $\sigma_V = 0.50$. What is its expected return under P ? Under P^* ?
 - c. Compute the two-year futures price and the two-year forward price. Why are they equal?
 - d. A *log contract* pays $\ln S_T$ at $T = 2$. Use (2) to compute its price today.
2. A pension fund holds a stock index at $S_0 = 100$ and buys a one-year European put with strike $K = 95$ as insurance. The index volatility is $\sigma = 0.25$ and $r = 0.04$. Use $N(0.49) = 0.6879$ and $N(0.24) = 0.5948$.
 - a. Compute d_1 and d_2 , and the risk-neutral probability that the put is exercised.
 - b. The put's payoff is a long position in a contract paying K if $S_T < K$ and a short position in a contract paying S_T if $S_T < K$. Price each contract using (4) and its analog for the put, and compute the put price.
 - c. Use put-call parity to price the call with the same strike, and check it with (5).
3. A firm defaults with historical intensity $\lambda = 0.02$ per year, bondholders lose $L = 0.5$ at default, and $r = 0.04$. Its bonds trade at a credit spread of 300 basis points.
 - a. Compute the expected loss rate, the risk-neutral intensity λ^* , and the jump η in the SDF at default.
 - b. Compute the price of a five-year zero-coupon bond with face value 1 before default, and compare it with a risk-free bond.
 - c. Compute the probability that the firm defaults within five years under P and under P^* .
 - d. Evaluate the density Z_5 in (10) on paths with no default and on paths with exactly one default by year five. Explain why default paths receive more weight under P^* .