

Class 10

Geometric Brownian Motion

A Brownian motion can turn negative, and its changes do not depend on its level. Stock prices are positive, and a stock at \$200 moves about twice as many dollars in a day as the same stock at \$100. What is roughly stable is the distribution of *returns*, so the standard model lets returns follow a Brownian motion with drift:

$$\frac{dS_t}{S_t} = \mu dt + \sigma dB_t, \quad \text{or} \quad dS_t = \mu S_t dt + \sigma S_t dB_t. \quad (1)$$

The process $\{S_t\}$ is a **geometric Brownian motion** (GBM). The constant μ is the **expected return** per unit of time and σ is the **volatility** of returns, both usually quoted per year. Over a short interval Δt , the return $\Delta S_t/S_t$ is approximately normal with mean $\mu\Delta t$ and variance $\sigma^2\Delta t$.

Solving for the Stock Price

Itô's lemma applied to $X_t = \ln S_t$, with $f'(S) = 1/S$, $f''(S) = -1/S^2$ and $(dS_t)^2 = \sigma^2 S_t^2 dt$, gives

$$d \ln S_t = \frac{dS_t}{S_t} - \frac{1}{2} \frac{(dS_t)^2}{S_t^2} = \left(\mu - \frac{1}{2} \sigma^2 \right) dt + \sigma dB_t.$$

The log price is a Brownian motion with drift, and integrating from 0 to T ,

$$S_T = S_0 \exp \left(\left(\mu - \frac{1}{2} \sigma^2 \right) T + \sigma B_T \right). \quad (2)$$

The term $-\frac{1}{2}\sigma^2$ is the **Itô correction**: ordinary calculus would wrongly give the log price a drift of μ . Equivalently, $S_t = S_0 e^{\mu t} Z_t$, where $Z_t = \exp \left(\sigma B_t - \frac{1}{2} \sigma^2 t \right)$ is the exponential martingale, with $dZ_t = \sigma Z_t dB_t$.

The Distribution of the Stock Price

Since $B_T \sim \mathcal{N}(0, T)$, (2) implies

$$\ln S_T \sim \mathcal{N}\left(\ln S_0 + \left(\mu - \frac{1}{2}\sigma^2\right)T, \sigma^2 T\right), \quad (3)$$

so S_T is **lognormal**. The continuously compounded return $\ln(S_T/S_0)$ is normal with mean $\left(\mu - \frac{1}{2}\sigma^2\right)T$ and standard deviation $\sigma\sqrt{T}$. To compute a probability about the price, translate it into a statement about this normal return and standardize. For instance,

$$P(S_T > K) = P\left(Z > \frac{\ln(K/S_0) - \left(\mu - \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}\right),$$

where Z is standard normal. Probability intervals for $\ln S_T$ become intervals for S_T by exponentiating the endpoints, and they are not symmetric around the current price.

Moments of the Stock Price

Because $S_t e^{-\mu t} = S_0 Z_t$ is a martingale, the expected price grows at the rate μ , and the normal MGF gives the second moment $E(S_T^2) = E(S_T)^2 e^{\sigma^2 T}$. Therefore,

$$E(S_T) = S_0 e^{\mu T}, \quad \text{SD}(S_T) = E(S_T) \sqrt{e^{\sigma^2 T} - 1}. \quad (4)$$

The uncertainty about the future price grows much faster than its expected value.

Expected Return and Growth Rate

The log price grows at the rate $\mu - \frac{1}{2}\sigma^2$, and the expected price at the faster rate μ . The gap comes from the right skew of the lognormal distribution. The median is the price at which the log price is at its mean,

$$\text{Median}(S_T) = S_0 e^{(\mu - \frac{1}{2}\sigma^2)T} < S_0 e^{\mu T} = E(S_T). \quad (5)$$

A few paths that end very high pull the mean above what a typical path achieves, as Figure 1 shows. The realized growth rate $\frac{1}{T} \ln(S_T/S_0)$ converges to $\mu - \frac{1}{2}\sigma^2$ as $T \rightarrow \infty$, so this is the rate a long-term holder typically earns, while μ is the rate at which wealth grows on average

across all paths. The price ends below its mean when $\sigma B_T < \frac{1}{2}\sigma^2 T$, which has probability $P\left(Z < \frac{1}{2}\sigma\sqrt{T}\right) > \frac{1}{2}$.

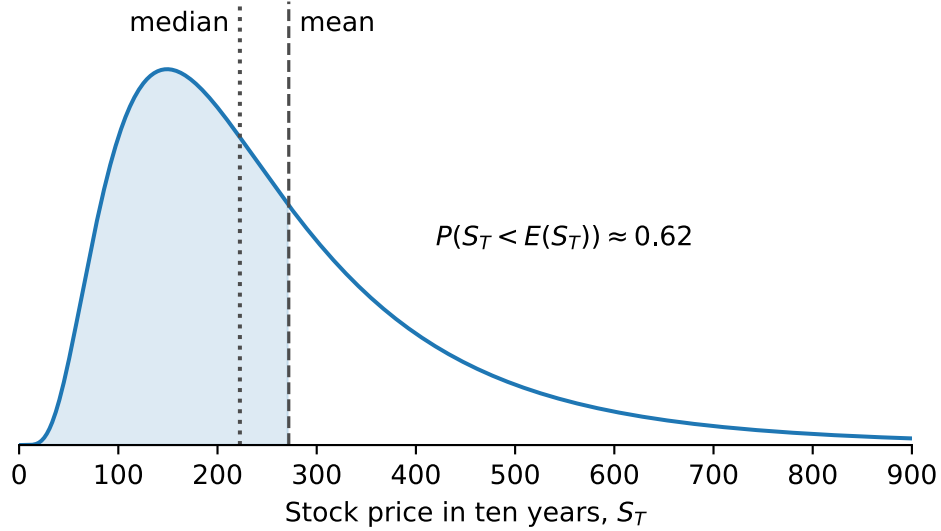


Figure 1: Density of the stock price in ten years for $S_0 = 100$, $\mu = 0.10$ and $\sigma = 0.20$. The dotted line marks the median $S_0 e^{(\mu - \frac{1}{2}\sigma^2)T} = 221.4$ and the dashed line the mean $S_0 e^{\mu T} = 271.8$.

Functions of the Stock Price

The price of a derivative depends on the stock price and time, $X_t = f(S_t, t)$. Substituting (1) and $(dS_t)^2 = \sigma^2 S_t^2 dt$ into Itô's lemma gives

$$dX_t = \left(\mu S_t X_S + \frac{1}{2} \sigma^2 S_t^2 X_{SS} + \frac{\partial f}{\partial t} \right) dt + \sigma S_t X_S dB_t, \quad (6)$$

with $X_S = \partial f / \partial S$ and $X_{SS} = \partial^2 f / \partial S^2$. The function is driven by the same Brownian motion as the stock, with a volatility set by its sensitivity X_S . This common risk is what lets an option be replicated with the stock. For a claim maturing at T it is convenient to use the time to maturity $\tau = T - t$, and since $\partial f / \partial t = -X_\tau$,

$$dX = X_S dS + \frac{1}{2} X_{SS} (dS)^2 - X_\tau dt.$$

A power of a GBM is again a GBM. For $X_t = S_t^a$,

$$\frac{dX_t}{X_t} = \left(a\mu + \frac{1}{2} a(a-1)\sigma^2 \right) dt + a\sigma dB_t. \quad (7)$$

With $a = -1$ this explains **Siegel's paradox**: if S_t is the dollar price of a euro, the euro price of a dollar, $1/S_t$, has expected rate of change $-\mu + \sigma^2$, so for $0 < \mu < \sigma^2$ each currency is expected to appreciate against the other.

Practice Problems

1. A broad stock index follows a GBM with an expected return of $\mu = 9\%$ per year and a volatility of $\sigma = 20\%$ per year.
 - a. Compute the probability that a buy-and-hold investor in the index loses money over one year, and over 25 years.
 - b. Compute the probability that the index at least doubles over ten years, and compare it with the expected value of S_{10}/S_0 .
2. An investor puts \$1 million into one of two funds and holds it for 20 years. Fund A has $\mu = 10\%$ and $\sigma = 30\%$, and fund B has $\mu = 8\%$ and $\sigma = 15\%$, both per year.
 - a. For each fund, compute the expected value and the median value of the investment after 20 years.
 - b. Which fund offers the higher expected return, and which the higher long-run realized growth rate? Explain the difference.
 - c. Compute the probability that the investment in fund A ends below the median value of the investment in fund B.
3. A stock follows a GBM with $\mu = 6\%$ and $\sigma = 30\%$ per year, and consider $X_t = S_t^a$ for a constant a .
 - a. Use (6) to derive the expected return and the volatility of X_t .
 - b. Find the value $a \neq 0$ for which X_t has zero drift.
4. The dollar price of one euro, S_t , follows a GBM with $\mu = 1\%$ and $\sigma = 8\%$ per year.
 - a. Compute the expected rate of change of the euro price of one dollar, $1/S_t$, and check whether the two expected rates of change agree on which currency is expected to appreciate.
 - b. For which values of μ , given $\sigma = 8\%$, is each currency expected to appreciate against the other?
 - c. Compute the drift of $\ln S_t$ and of $\ln(1/S_t)$, and explain why the log changes do not produce a paradox.