

Class 8

Trading Gains and the Itô Integral

Throughout, $\{B_t\}$ is a Brownian motion and $\mathcal{F}_t$ is the information generated by its path up to time t . A process $\{b_t\}$ is **adapted** if b_t is known given $\mathcal{F}_t$.

An investor who holds b_{t_j} units of an asset whose price is a Brownian motion from t_j to t_{j+1} , choosing each position before seeing the next price change, earns

$$I_t = \sum_{j=0}^{n-1} b_{t_j} (B_{t_{j+1}} - B_{t_j}). \quad (1)$$

The holding is evaluated at the *left* endpoint of each period. For a general adapted integrand, the **Itô integral** is the mean-square limit of these sums as the partition gets finer,

$$\int_0^t b_s dB_s = \lim_{\|\Pi\| \rightarrow 0} \sum_{j=0}^{n-1} b_{t_j} (B_{t_{j+1}} - B_{t_j}). \quad (2)$$

Because Brownian paths have infinite total variation, the evaluation point matters. Summing $B_{t_j} \Delta B_j = \frac{1}{2}(B_{t_{j+1}}^2 - B_{t_j}^2) - \frac{1}{2}(\Delta B_j)^2$, the first term telescopes and the second converges to half the quadratic variation, so

$$\int_0^T B_s dB_s = \frac{1}{2}(B_T^2 - T). \quad (3)$$

The right endpoint would give $\frac{1}{2}(B_T^2 + T)$, the gains of an investor who picks each position after seeing the price move.

Properties of the Itô Integral

Each increment ΔB_j has mean zero and is independent of $\mathcal{F}_{t_j}$, while b_{t_j} is known at t_j . From this single fact, if $E\left(\int_0^t b_s^2 ds\right) < \infty$, then $I_t = \int_0^t b_s dB_s$ satisfies:

1. **Martingale.** $\{I_t\}$ is a martingale with continuous paths, so $E(I_t) = 0$.
2. **Itô isometry.** $E(I_t^2) = E\left(\int_0^t b_s^2 ds\right)$.
3. **Quadratic variation.** $[I, I]_t = \int_0^t b_s^2 ds$.

A position of b_s units adds $b_s^2 ds$ to the variance of the gains, and no strategy based on past information turns a fair game into a favorable one. When the integrand is a deterministic function $f(s)$, each approximating sum is a sum of independent normals, so

$$\int_0^t f(s) dB_s \sim \mathcal{N}\left(0, \int_0^t f(s)^2 ds\right). \quad (4)$$

If we only require $\int_0^t b_s^2 ds < \infty$ path by path, the integral still has zero drift but may fail to be a martingale: it is a **local martingale**.

Itô Processes

An **Itô process** combines a time integral and an Itô integral,

$$X_t = X_0 + \int_0^t a_s ds + \int_0^t b_s dB_s, \quad dX_t = a_t dt + b_t dB_t, \quad (5)$$

with adapted **drift** a_t and **volatility** b_t . Over a short interval, $E(dX_t | \mathcal{F}_t) = a_t dt$ and $V(dX_t | \mathcal{F}_t) = b_t^2 dt$. Drift and volatility are determined by the paths, so an Itô process is a martingale only if its drift is zero. With the multiplication table $(dt)^2 = 0$, $(dt)(dB_t) = 0$ and $(dB_t)^2 = dt$,

$$(dX_t)^2 = b_t^2 dt, \quad [X, X]_t = \int_0^t b_s^2 ds.$$

The drift moves X by amounts of order dt and leaves no trace in the quadratic variation.

Itô's Formula

If $f(x, t)$ is twice continuously differentiable in x and once in t , then $Y_t = f(X_t, t)$ is an Itô process with

$$dY_t = \frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial x} dX_t + \frac{1}{2} \frac{\partial^2 f}{\partial x^2} (dX_t)^2. \quad (6)$$

Substituting dX_t and $(dX_t)^2 = b_t^2 dt$,

$$dY_t = \left(\frac{\partial f}{\partial t} + a_t \frac{\partial f}{\partial x} + \frac{1}{2} b_t^2 \frac{\partial^2 f}{\partial x^2} \right) dt + b_t \frac{\partial f}{\partial x} dB_t. \quad (7)$$

The term $\frac{1}{2} b_t^2 \frac{\partial^2 f}{\partial x^2} dt$ is the **Itô correction**. In practice, expand f to second order in dX and first order in dt , then simplify with the multiplication table. Three cases with $X_t = B_t$:

- $d(B_t^2) = 2B_t dB_t + dt$, which recovers (3) in one line. The drift of B_t^2 is one, so $B_t^2 - t$ is a martingale, as Figure 1 shows.
- $Z_t = \exp\left(\theta B_t - \frac{1}{2}\theta^2 t\right)$ satisfies $dZ_t = \theta Z_t dB_t$: the Itô correction exactly offsets $-\frac{1}{2}\theta^2 t$, so Z_t is a martingale.
- $d(tB_t) = B_t dt + t dB_t$ has no correction, and gives $\int_0^T B_s ds = \int_0^T (T-s) dB_s \sim \mathcal{N}(0, T^3/3)$.

Since Itô integrals have mean zero, taking expectations of the integral form of Itô's formula leaves only the drift: $E(Y_t) = Y_0 + E\left(\int_0^t \text{drift}_s ds\right)$.

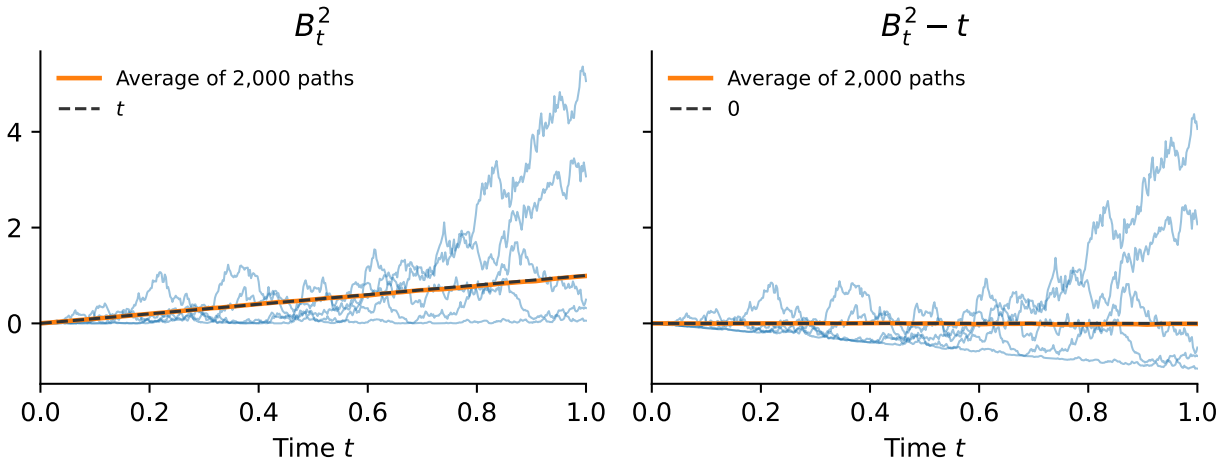


Figure 1: Five simulated paths of B_t^2 and of $B_t^2 - t$ on $[0, 1]$, with the average over 2,000 paths. Itô's formula gives B_t^2 a drift of one, so its average grows like t , while subtracting t removes the drift and leaves a martingale whose average stays at zero.

Practice Problems

1. The price of an asset follows a Brownian motion B_t , measured in dollars. A trader holds one unit from time 0 to time 1. At time 1 the trader looks at the price and, for the period from 1 to 2, holds two units if $B_1 > 0$ and nothing otherwise. Let I_2 be the trading gains at time 2.
 - a. Write I_2 as a sum of the form (1) and explain why the strategy is adapted.
 - b. Compute $E(I_2)$ and $E(I_2 \mid \mathcal{F}_1)$.
 - c. Use the Itô isometry to compute $V(I_2)$.
 - d. A second trader holds one unit from 0 to 1 and, for the period from 1 to 2, holds two units if $B_2 > B_1$ and nothing otherwise. Compute the expected gains of this trader and explain why this strategy is not allowed.
2. A fund manager holds two units of an asset whose price follows a Brownian motion during the first half of the year and cuts the position to one unit for the second half. The gains over the year are $I = \int_0^1 f(s) dB_s$, with $f(s) = 2$ for $s < \frac{1}{2}$ and $f(s) = 1$ for $s \geq \frac{1}{2}$.
 - a. Find the distribution of I .
 - b. Compute $P(I > 1)$.
 - c. Compute $\text{Cov}(I, B_1)$ and the correlation between the gains and the price change over the year.
3. Let X_t follow $dX_t = -\frac{1}{2}\kappa X_t dt + \frac{1}{2}\sigma dB_t$, with constants $\kappa > 0$ and $\sigma > 0$, and model the short-term interest rate as $r_t = X_t^2$.
 - a. Use Itô's formula to show that $dr_t = \kappa(\theta - r_t) dt + \sigma X_t dB_t$, and find θ in terms of κ and σ .
 - b. Show that $(dr_t)^2 = \sigma^2 r_t dt$.
 - c. With $\kappa = 0.5$ and $\sigma = 0.2$, compute θ . When $r_t = 4\%$, compute the drift of the interest rate and the instantaneous standard deviation of dr_t per unit of $\sqrt{dt}$.
4. Itô's formula is also a tool for computing moments and finding martingales.
 - a. Apply Itô's formula to B_t^4 , take expectations, and show that $E(B_t^4) = 3t^2$. Repeat the argument with B_t^6 to compute $E(B_t^6)$.
 - b. Show that $M_t = e^{t/2} \cos(B_t)$ has zero drift, and use this to compute $E(\cos(B_t))$.