

## Class 5

### Martingales

A process  $\{M_n\}$  is **adapted** to a filtration  $\{\mathcal{F}_n\}$  if  $M_n$  is known given  $\mathcal{F}_n$ . It is a **martingale** if, in addition, for all  $n$ ,

$$E(M_{n+1} | \mathcal{F}_n) = M_n. \quad (1)$$

Given everything known at time  $n$ , the best forecast of the next value is the current value: the process does not drift up or down. By the tower property, a martingale also has  $E(M_n) = E(M_0)$  for every  $n$ .

### Three Families of Martingales

Let  $X_1, X_2, \dots$  be independent and let  $\mathcal{F}_n$  be generated by  $X_1, \dots, X_n$ . The variables need not be identically distributed. Each family below follows from taking out what is known and replacing  $E(\cdot | \mathcal{F}_n)$  by  $E(\cdot)$  for functions of  $X_{n+1}$  alone.

1. **Sums.** If  $E(X_n) = 0$ , then  $S_n = S_0 + \sum_{i=1}^n X_i$  is a martingale.
2. **Squares.** If, in addition,  $V(X_n) = \sigma^2$ , then  $M_n = S_n^2 - n\sigma^2$  is a martingale. Expanding  $(S_n + X_{n+1})^2$ , the cross term has conditional mean zero and  $X_{n+1}^2$  adds  $\sigma^2$  on average, which the term  $-n\sigma^2$  removes.
3. **Products.** If  $X_n > 0$  and  $E(X_n) = 1$ , then  $M_n = X_1 X_2 \cdots X_n$ , with  $M_0 = 1$ , is a martingale, since  $E(M_n X_{n+1} | \mathcal{F}_n) = M_n E(X_{n+1})$ .

### Betting Strategies

Let  $M_k - M_{k-1}$  be the gain from a one-dollar bet on round  $k$ , and let the gambler stake  $H_k$  dollars on that round. The stake is chosen before the round is played, so  $H_k$  is known given  $\mathcal{F}_{k-1}$ : the strategy is **predictable**. Starting from  $W_0$ , wealth after  $n$  rounds is the **martingale transform**

$$W_n = W_0 + \sum_{k=1}^n H_k(M_k - M_{k-1}). \quad (2)$$

In finance,  $M_k$  is a (discounted) price,  $H_k$  is the number of shares held from  $k - 1$  to  $k$ , and  $W_n$  is the value of a self-financing strategy.

#### Betting Strategies Cannot Beat a Martingale

If  $\{M_n\}$  is a martingale and  $\{H_k\}$  is predictable with each  $H_k$  bounded, then  $\{W_n\}$  is a martingale and  $E(W_n) = W_0$  for every  $n$ .

The proof is one line:  $E(W_{n+1} | \mathcal{F}_n) = W_n + H_{n+1}(E(M_{n+1} | \mathcal{F}_n) - M_n) = W_n$ . In continuous time, these sums become the Itô integral.

The **doubling strategy** bets one dollar on a fair coin, doubles the stake after every loss, and stops at the first win. If the first win comes on round  $k$ , the gambler has lost  $2^{k-1} - 1$  dollars and then wins  $2^{k-1}$ , a net gain of one dollar, as in Figure 1. A win comes with probability one, yet over any  $n$  rounds  $E(W_n) = (1 - 2^{-n}) - 2^{-n}(2^n - 1) = 0$ . The sure profit is paid for by an unbounded line of credit.

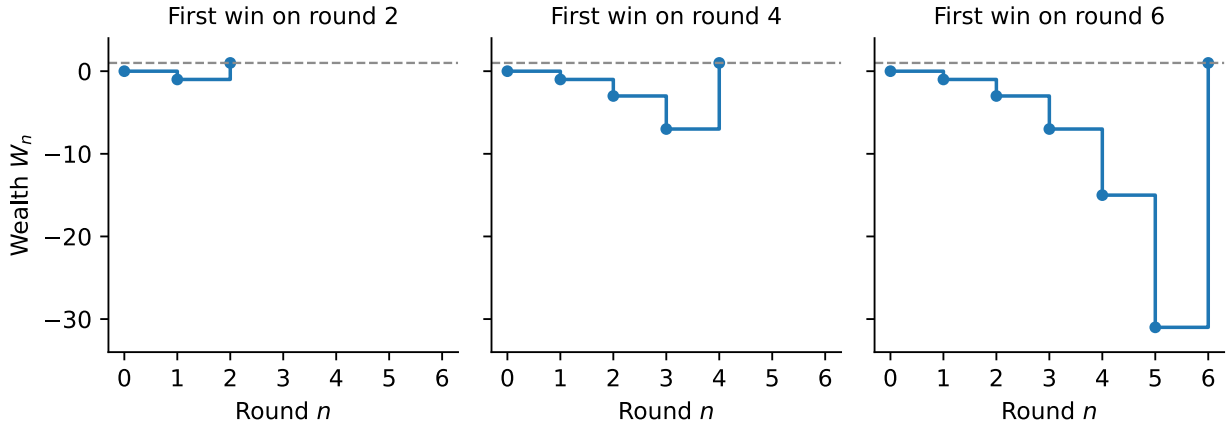


**Figure 1:** Wealth of the doubling strategy when the first win comes on round 2, 4 or 6. Every path ends one dollar ahead, but the losses before the win double with each round.

## Stopping Times and Stopped Martingales

A random time  $\tau$  with values in  $\{0, 1, 2, \dots\} \cup \{\infty\}$  is a **stopping time** if  $\{\tau \leq n\} \in \mathcal{F}_n$  for every  $n$ : whether we have stopped by time  $n$  is known at time  $n$ . The truncated time  $n \wedge \tau = \min\{n, \tau\}$

is always finite, and the **stopped process**  $M_{n \wedge \tau}$  freezes  $M$  at time  $\tau$ . Stopping is itself a betting strategy, one dollar per round while the game runs and nothing afterwards:

$$M_{n \wedge \tau} = M_0 + \sum_{k=1}^n \mathbf{1}_{\{\tau \geq k\}} (M_k - M_{k-1}). \quad (3)$$

The stake  $\mathbf{1}_{\{\tau \geq k\}}$  is predictable because  $\{\tau \geq k\}$  is the complement of  $\{\tau \leq k-1\} \in \mathcal{F}_{k-1}$ , so a stopped martingale is a martingale and  $E(M_{n \wedge \tau}) = E(M_0)$  for every  $n$ .

## Optional Stopping

If  $\tau$  is finite,  $M_{n \wedge \tau} \rightarrow M_\tau$  as  $n \rightarrow \infty$ , but the expectations need not converge: the doubling strategy has  $W_\tau = 1$  and  $W_0 = 0$ .

### Optional Stopping Theorem

Let  $\{M_n\}$  be a martingale and  $\tau$  a stopping time with  $P(\tau < \infty) = 1$ . If there is a random variable  $Y$  with  $E(Y) < \infty$  such that  $|M_{n \wedge \tau}| \leq Y$  for all  $n$ , then

$$E(M_\tau) = E(M_0). \quad (4)$$

In particular, the result holds when the stopped process is bounded by a constant.

The doubling strategy fails the condition: the wealth just before the win is  $-(2^{\tau-1} - 1)$ , and  $E(2^{\tau-1}) = \sum_k 2^{k-1} 2^{-k} = \infty$ .

## Ruin Probabilities Again

For the fair random walk with  $S_0 = 0$  and  $\tau = \min\{n > 0 : S_n = A \text{ or } S_n = -B\}$ , the stopped walk is bounded by  $\max(A, B)$ , so optional stopping gives  $E(S_\tau) = 0$ , that is,  $A P(S_\tau = A) - B(1 - P(S_\tau = A)) = 0$  and  $P(S_\tau = A) = B/(A + B)$ . The martingale  $S_n^2 - n$  is dominated by  $\max(A^2, B^2) + \tau$ , which is integrable because  $E(\tau) < \infty$ , so  $E(S_\tau^2) = E(\tau)$ , which gives  $E(\tau) = AB$ .

With a biased coin,  $S_n$  is not a martingale, but each factor  $(q/p)^{X_i}$  is positive with mean  $p \frac{q}{p} + q \frac{p}{q} = 1$ , so the product  $(q/p)^{S_n}$  is a martingale, bounded between  $(q/p)^{-B}$  and  $(q/p)^A$  when stopped. Optional stopping, together with the martingale  $S_n - n(p - q)$ , gives

$$P_A = P(S_\tau = A) = \frac{(q/p)^B - 1}{(q/p)^{A+B} - 1}, \quad E(\tau) = \frac{AP_A - B(1 - P_A)}{p - q}. \quad (5)$$

The same pattern returns with Poisson processes and Brownian motion: an exponential martingale delivers the ruin probability.

## Practice Problems

1. Let  $X_1, X_2, \dots$  be independent fair coin flips, equal to 1 or  $-1$  with probability  $1/2$ , let  $S_0 = 0$ ,  $S_n = X_1 + \dots + X_n$ , and let  $\mathcal{F}_n$  be generated by  $X_1, \dots, X_n$ .
  - a. Compute  $E(S_{n+1}^2 | \mathcal{F}_n)$ . Is  $\{S_n^2\}$  a martingale?
  - b. Show that  $\{S_n^3 - 3nS_n\}$  is a martingale.
  - c. Find the constant  $c$  such that  $\{c^n 2^{S_n}\}$  is a martingale.
2. A stock trades at a price  $M_0 = 100$  that is a martingale. Each period the price rises or falls by \$10 with probability  $1/2$ , for two periods. A trader buys one share at time 0. If the price falls in the first period, the trader “buys the dip” and holds two shares over the second period; if it rises, the trader sells and holds nothing over the second period. The trader starts with  $W_0 = 0$  and borrows at zero interest to buy.
  - a. Write down  $H_1$  and  $H_2$  and explain why the strategy is predictable.
  - b. Compute  $W_2$  on each of the four price paths, the probability that the trader makes money, and  $E(W_2)$ .
  - c. Compute  $E(W_2 | \mathcal{F}_1)$  after a fall in the first period, and compare it with  $W_1$ .
3. A market maker’s inventory changes each period by  $+1, 0$  or  $-1$  with probability  $1/3$  each, independently, starting from zero. The market maker hedges as soon as the inventory reaches  $+4$  or  $-2$ . Assume  $\tau$  is finite with  $E(\tau) < \infty$ .
  - a. Show that the inventory  $S_n$  is a martingale and find  $\sigma^2$  such that  $S_n^2 - n\sigma^2$  is a martingale.
  - b. Use optional stopping to compute the probability that the inventory reaches  $+4$  first.
  - c. Use optional stopping to compute the expected number of periods until the market maker hedges. Explain why it differs from  $AB = 8$ .
4. A trader’s strategy gains \$1 with probability  $p = 0.6$  and loses \$1 with probability  $q = 0.4$  on each trade, independently. The trader stops after making \$3 or after losing \$3.
  - a. Between which two values does the stopped martingale  $(q/p)^{S_{n \wedge \tau}}$  lie? Why does this matter for optional stopping?
  - b. Use optional stopping to compute the probability that the trader makes \$3 first.
  - c. Use the martingale  $S_n - n(p - q)$  to compute the expected number of trades.