

Class 4

The Gambler's Problem

A gambler bets one dollar on each flip of a fair coin. The flips $X_1, X_2, \dots$ are independent and equal to 1 or -1 with probability $\frac{1}{2}$ each, so they have mean zero and variance one, and the gambler's wealth is the random walk

$$S_n = S_0 + \sum_{i=1}^n X_i.$$

The gambler stops the first time the wealth reaches a target A or a floor $-B$, where A and B are positive integers:

$$\tau = \min\{n > 0 : S_n = A \text{ or } S_n = -B\}.$$

We want two numbers: the probability $P(S_\tau = A)$ of winning A dollars before losing B , and the expected duration $E(\tau)$ of the game.

The Game Ends

Both questions make sense only if the game ends. From any position strictly between $-B$ and A , a streak of $A + B$ wins reaches A . Split time into blocks of $A + B$ flips. Each block is a winning streak with probability $p = (1/2)^{A+B} > 0$, independently of the other blocks, and the game can still be running after n blocks only if none of them was a streak. Therefore,

$$P(\tau > n(A + B)) \leq (1 - p)^n \xrightarrow{n \rightarrow \infty} 0, \quad (1)$$

so $P(\tau = \infty) = 0$. Writing $E(\tau) = \sum_{m=0}^{\infty} P(\tau > m)$, the same bound gives $E(\tau) \leq (A + B)/p < \infty$, and in fact every moment of τ is finite. The bound is crude, but it shows the questions have answers.

The Probability of Winning

The game has no clock, so the only state variable is the current wealth. Let $f(K) = P(S_\tau = A \mid S_0 = K)$ for $-B \leq K \leq A$. Conditioning on the first flip, the gambler moves to $K + 1$ or $K - 1$ and the problem starts over from there:

$$f(K) = \frac{1}{2}f(K-1) + \frac{1}{2}f(K+1), \quad f(-B) = 0, \quad f(A) = 1. \quad (2)$$

This is a second-order linear difference equation. Trying $f(K) = z^K$ gives $(z-1)^2 = 0$, a repeated root, so the general solution is $f(K) = \alpha + \beta K$. The boundary conditions pin down the constants:

$$f(K) = \frac{B+K}{A+B}, \quad P(S_\tau = A \mid S_0 = 0) = \frac{B}{A+B}. \quad (3)$$

A trade with a take-profit at $+A$ and a stop-loss at $-B$ on a fair random walk therefore has expected profit $A \frac{B}{A+B} - B \frac{A}{A+B} = 0$. The exit levels change the shape of the outcome, frequent small gains against rare large losses or the reverse, but not its mean.

Equations like (2) appear whenever a quantity depends only on the current state and the system moves to neighboring states with known probabilities. Pricing an option in a binomial tree works the same way.

The Expected Duration

Let $g(K) = E(\tau \mid S_0 = K)$. One flip adds one period and moves the gambler to a neighboring state, so

$$g(K) = 1 + \frac{1}{2}g(K-1) + \frac{1}{2}g(K+1), \quad g(-B) = g(A) = 0. \quad (4)$$

The homogeneous part is the same as before, and trying γK^2 gives the particular solution $-K^2$. Imposing the boundary conditions on $\alpha + \beta K - K^2$,

$$g(K) = AB + (A-B)K - K^2, \quad E(\tau \mid S_0 = 0) = AB. \quad (5)$$

Doubling both barriers quadruples the expected duration. A random walk needs about n^2 steps to travel a distance n , since its typical distance after m steps is of order $\sqrt{m}$: the scaling that turns a random walk into Brownian motion.

A Biased Coin

If each bet is won with probability p and lost with probability $q = 1 - p$, with $p \neq q$, the recursion becomes $f(K) = pf(K + 1) + qf(K - 1)$. Trying z^K now gives $pz^2 - z + q = 0$, with distinct roots $z = 1$ and $z = q/p$, so $f(K) = \alpha + \beta(q/p)^K$ and

$$f(K) = \frac{(q/p)^{K+B} - 1}{(q/p)^{A+B} - 1}. \quad (6)$$

As $p \rightarrow 1/2$ this converges to the fair answer $(K + B)/(A + B)$. Figure 1 shows how fast a small edge compounds: the gambler keeps betting until a barrier is reached, so the house edge accumulates over the whole game.

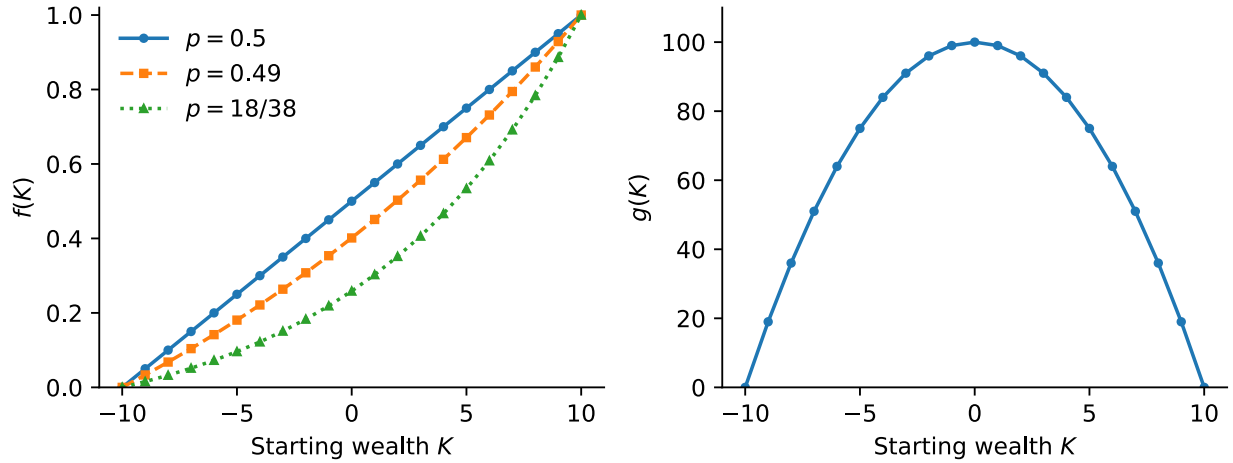


Figure 1: Left: probability $f(K)$ of reaching $A = 10$ before $-B = -10$ from wealth K , for three probabilities p of winning each bet. Right: expected duration $g(K)$ for the fair coin.

Practice Problems

1. A stock price moves up or down by \$1 each period with probability $1/2$. A trader buys one share and places a take-profit order \$8 above the purchase price and a stop-loss order \$4 below it.
 - a. Compute the probability that the trade closes with the profit and the expected profit of the trade.
 - b. Compute the expected number of periods until the trade closes.
 - c. A few periods later the price is \$2 above the purchase price and neither order has been executed. Compute the probability that the trade now closes with the profit and the expected number of remaining periods.
 - d. Repeat the previous part if the price is instead \$2 below the purchase price.
2. A thinly traded stock does not trade in half of the periods. Each period its price rises by \$1 with probability $1/4$, falls by \$1 with probability $1/4$, and stays unchanged with probability $1/2$. A trader exits when the price is \$6 above or \$3 below the entry price. Let $f(K)$ and $g(K)$ be the probability of exiting with the profit and the expected number of periods to exit when the price is K dollars from the entry price.
 - a. Condition on the first period to write the difference equations for $f(K)$ and $g(K)$, with their boundary conditions.
 - b. Show that $f(K)$ is the same as for the fair coin, and compute $f(0)$.
 - c. Show that $g(K) = 2(AB + (A - B)K - K^2)$ with $A = 6$ and $B = 3$, and compute $g(0)$. Explain the factor of two.
3. A trading strategy gains \$1 with probability $p = 0.52$ and loses \$1 with probability $q = 0.48$ on each trade, independently. The trader has \$50 of risk capital and stops after losing it or after making \$50.
 - a. Compute the probability of making \$50 before losing \$50 when the trader risks \$1 per trade.
 - b. Compute the same probability when the trader risks \$5 per trade, so that each trade gains or loses \$5 with the same probabilities.
 - c. Compare your answers with the fair coin. Should a player with an edge make small or large bets?