Prof. Lorenzo Naranjo

Stochastic Calculus Table

The Itô Integral

For an adapted process b_t with $\mathrm{E}\left(\int_0^t b_s^2 \, \mathrm{d}s\right) < \infty$, the integral $I_t = \int_0^t b_s \, \mathrm{d}B_s$ is a martingale with

$$\mathrm{E}(I_t) = 0, \qquad \mathrm{E}\left(I_t^2\right) = \mathrm{E}\left(\int_0^t b_s^2 \, \mathrm{d}s\right), \qquad [I, I]_t = \int_0^t b_s^2 \, \mathrm{d}s.$$

For a deterministic integrand, $\int_0^t f(s) \, \mathrm{d}B_s \sim \mathcal{N}\left(0, \int_0^t f(s)^2 \, \mathrm{d}s\right)$.

Itô Processes

An Itô process satisfies $\mathrm{d}X_t = a_t \, \mathrm{d}t + b_t \, \mathrm{d}B_t$, with drift a_t and volatility b_t. It is a martingale only if its drift is zero. Products of increments follow the table below, where the increments of B_1 and B_2 have correlation ρ.

Product	$\mathrm{d}t$	$\mathrm{d}B_1$	$\mathrm{d}B_2$
$\mathrm{d}t$	0	0	0
$\mathrm{d}B_1$	0	$\mathrm{d}t$	$\rho \, \mathrm{d}t$
$\mathrm{d}B_2$	0	$\rho \, \mathrm{d}t$	$\mathrm{d}t$

For $\mathrm{d}X_t = a_t \, \mathrm{d}t + b_t \, \mathrm{d}B_1$ and $\mathrm{d}Y_t = c_t \, \mathrm{d}t + d_t \, \mathrm{d}B_2$, the table gives

$$(\mathrm{d}X_t)^2 = b_t^2 \, \mathrm{d}t, \qquad (\mathrm{d}X_t)(\mathrm{d}Y_t) = \rho \, b_t d_t \, \mathrm{d}t, \qquad [X, X]_t = \int_0^t b_s^2 \, \mathrm{d}s.$$

Itô's Formula

For $Y_t = f(X_t, t)$,

$$\mathrm{d}Y_t = \frac{\partial f}{\partial t} \mathrm{d}t + \frac{\partial f}{\partial x} \mathrm{d}X_t + \frac{1}{2} \frac{\partial^2 f}{\partial x^2} (\mathrm{d}X_t)^2 = \left(\frac{\partial f}{\partial t} + a_t \frac{\partial f}{\partial x} + \frac{1}{2} b_t^2 \frac{\partial^2 f}{\partial x^2} \right) \mathrm{d}t + b_t \frac{\partial f}{\partial x} \mathrm{d}B_t.$$

For a function $f(x, y, t)$ of two Itô processes, with the product rule as the case $f(x, y) = xy$,

$$\mathrm{d}f = \frac{\partial f}{\partial t} \mathrm{d}t + \frac{\partial f}{\partial x} \mathrm{d}X + \frac{\partial f}{\partial y} \mathrm{d}Y + \frac{1}{2} \frac{\partial^2 f}{\partial x^2} (\mathrm{d}X)^2 + \frac{\partial^2 f}{\partial x \partial y} (\mathrm{d}X)(\mathrm{d}Y) + \frac{1}{2} \frac{\partial^2 f}{\partial y^2} (\mathrm{d}Y)^2,$$

$$\mathrm{d}(X_t Y_t) = X_t \, \mathrm{d}Y_t + Y_t \, \mathrm{d}X_t + (\mathrm{d}X_t)(\mathrm{d}Y_t), \qquad \frac{\mathrm{d}(X_t Y_t)}{X_t Y_t} = \frac{\mathrm{d}X_t}{X_t} + \frac{\mathrm{d}Y_t}{Y_t} + \frac{\mathrm{d}X_t}{X_t} \frac{\mathrm{d}Y_t}{Y_t}.$$

Girsanov's Theorem

Let $Z_T = \exp\left(-\theta B_T - \frac{1}{2}\theta^2 T\right)$ and $\mathrm{P}^*(A) = \mathrm{E}\left(Z_T \mathbb{1}_{\{A\}}\right)$. Under P^*, the process $B_t^* = B_t + \theta t$ is a Brownian motion, so

$$\mathrm{d}B_t = \mathrm{d}B_t^* - \theta\,\mathrm{d}t, \qquad \mathrm{d}X_t = (a_t - \theta b_t)\,\mathrm{d}t + b_t\,\mathrm{d}B_t^*.$$

The change of measure shifts the drift by $-\theta b_t$ and leaves the volatility unchanged.

Geometric Brownian Motion

The stock price follows $\mathrm{d}S_t/S_t = \mu\,\mathrm{d}t + \sigma\,\mathrm{d}B_t$. Then

$$\mathrm{d}\ln S_t = \left(\mu - \tfrac{1}{2}\sigma^2\right)\mathrm{d}t + \sigma\,\mathrm{d}B_t, \qquad S_T = S_0 \exp\left(\left(\mu - \tfrac{1}{2}\sigma^2\right)T + \sigma B_T\right),$$

so that $\ln S_T \sim \mathcal{N}\left(\ln S_0 + \left(\mu - \tfrac{1}{2}\sigma^2\right)T, \sigma^2 T\right)$, and

$$\mathrm{E}(S_T) = S_0 e^{\mu T}, \qquad \mathrm{SD}(S_T) = \mathrm{E}(S_T)\sqrt{e^{\sigma^2 T} - 1}, \qquad \text{Median}(S_T) = S_0 e^{\left(\mu - \frac{1}{2}\sigma^2\right)T}.$$

For $X_t = f(S_t, t)$,

$$\mathrm{d}X_t = \left(\mu S_t \frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 f}{\partial S^2} + \frac{\partial f}{\partial t}\right)\mathrm{d}t + \sigma S_t \frac{\partial f}{\partial S}\,\mathrm{d}B_t.$$

A power of the stock price is again a geometric Brownian motion:

$$\frac{\mathrm{d}(S_t^a)}{S_t^a} = \left(a\mu + \tfrac{1}{2}a(a-1)\sigma^2\right)\mathrm{d}t + a\sigma\,\mathrm{d}B_t.$$

Poisson Process and Brownian Motion

Property	Poisson process N_t	Brownian motion B_t
Increment over $(s, t]$	Poisson, mean $\lambda(t-s)$	Normal, variance $t-s$
Paths	Step functions, jumps of size one	Continuous, nowhere differentiable
Martingale	$N_t - \lambda t$	B_t
Second martingale	$(N_t - \lambda t)^2 - \lambda t$	$B_t^2 - t$
Exponential martingale	$e^{\theta N_t - \lambda t(e^\theta - 1)}$	$e^{\theta B_t - \frac{1}{2}\theta^2 t}$
Quadratic variation	N_t (random)	t (deterministic)