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Where Does cos(B) Spend Its Time?

The Question

The process $\cos(B_t)$ starts at 1 and stays in $[-1, 1]$. Its mean is the real part of the characteristic function of $B_t \sim \mathcal{N}(0, t)$ (Class 3),

$$\mathrm{E}(e^{iB_t}) = e^{-t/2}, \qquad \text{so} \qquad \mathrm{E}(\cos B_t) = e^{-t/2} \to 0. \tag{1}$$

Itô's formula (Class 8) gives the same answer: $M_t = e^{t/2} \cos B_t$ has $\mathrm{d}M_t = -e^{t/2} \sin B_t \, \mathrm{d}B_t$, so M is a martingale and $\mathrm{E}(\cos B_t) = e^{-t/2} \, \mathrm{E}(M_t) = e^{-t/2}$. Does a mean of zero mean that $\cos B_t$ settles near zero? We simulate 5,000 paths and find the opposite.

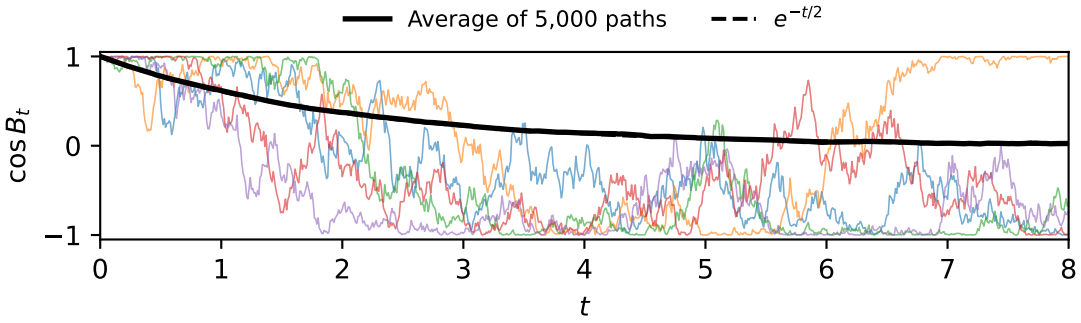


Figure 1: Five simulated paths of $\cos B_t$, the average over 5,000 paths, and the exact mean $e^{-t/2}$.

Where the Cosine Spends Its Time

Think of e^{iB_t} as a point on the unit circle with angle B_t. As t grows, the angle modulo 2π becomes uniform, and if Θ is uniform then $\mathrm{P}(\cos\Theta \le x) = 1 - \arccos(x)/\pi$, the **arcsine law** with density

$$g(x) = \frac{1}{\pi\sqrt{1 - x^2}}, \qquad -1 < x < 1. \tag{2}$$

The density is lowest at 0 and blows up at ± 1: the cosine is flat at the top and bottom of the circle and steep at the sides, so it lingers near ± 1 and rushes through 0. The circle is compact, so the same law also describes the fraction of time a single path spends in each region.

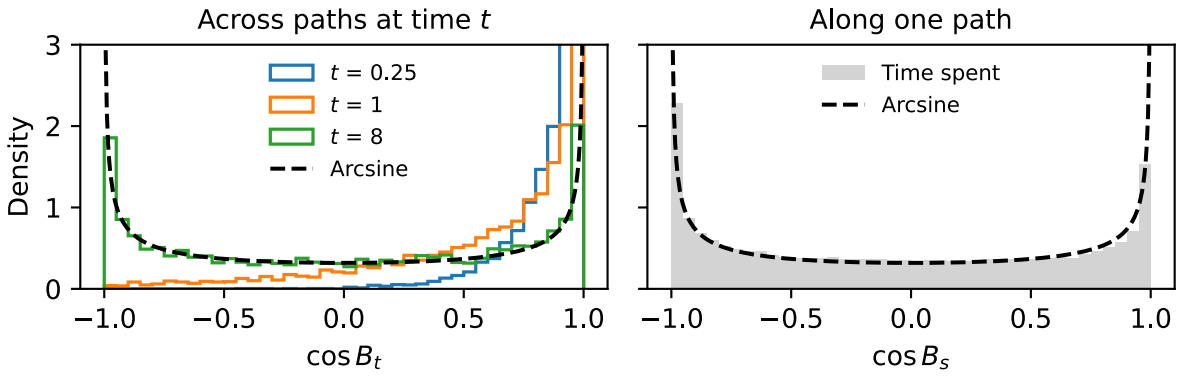


Figure 2: Left: distribution of $\cos B_t$ across 5,000 paths at three dates. Right: fraction of time one path spends in each bin over $0 \le s \le 1{,}000$. The dashed line is the arcsine density (2).

Along that path, $|\cos B_s| > 0.9$ for 27% of the time and $|\cos B_s| < 0.1$ for only 7%, two sets of the same length; the arcsine law gives 29% and 6%. The mean is zero because the visits to $+1$ and -1 cancel, not because the cosine sits near 0.

The Path on the Cylinder

Plotting e^{iB_t} against time gives the curve $(t, \cos B_t, \sin B_t)$ on a cylinder of radius one (Figure 3). The path winds back and forth at random and never leaves the surface, since $|e^{iB_t}| = 1$. The mean $e^{-t/2}$ from (1) heads straight for the axis, because the paths spread evenly around the circle.

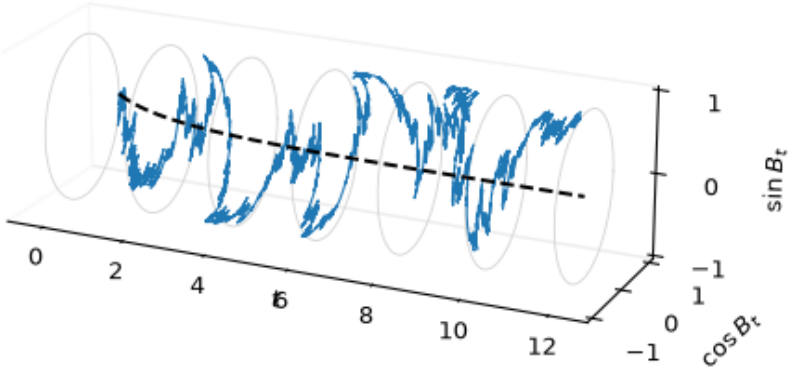


Figure 3: One path of $(t, \cos B_t, \sin B_t)$ on the unit cylinder and the mean $e^{-t/2}$ (dashed).