Stochastic Foundations for Finance — Prof. Lorenzo Naranjo
FIN 5380 — Fall 2025

Problem Set 5

Instructions: This problem set is due on 12/10 at 11:59 pm CST and is an individual assignment. All problems must be handwritten. Write full sentences to answer questions that require an explanation. Scan your work and submit a PDF file.

Problem 1 (A Random Walk). Let $\{X_n\}$ be a sequence of independent random variables, each taking values 1 or -1 with equal probability $1/2$. For $n \geq 1$, define

$$S_n = S_0 + \sum_{i=1}^{n} X_i,$$

and

$$\tau = \min\{n > 0 : S_n = A \text{ or } S_n = -B\}.$$

In the following, assume that $P(\tau = \infty \mid S_0 = 0) = 0$.

a. Compute $P(S_\tau = A \mid S_0 = 0)$ by defining a recursion on $f(k) = P(S_\tau = A \mid S_0 = k)$.
b. Compute $\mathrm{E}(\tau \mid S_0 = 0)$ by defining a recursion on $g(k) = \mathrm{E}(\tau \mid S_0 = k)$.

Problem 2 (Three Martingales). A martingale is a stochastic process $\{M_n\}$ that satisfies

$$\mathrm{E}(M_n \mid \mathcal{F}_{n-1}) = M_{n-1},$$

where $\mathcal{F}_{n-1}$ represents all information available up to time $n-1$. In the following, $\mathcal{F}_n$ represents the information set generated by the sequence $\{X_0, X_1, \dots, X_n\}$.

a. Let $\{X_n\}$ be a sequence of independent random variables such that $\mathrm{E}(X_n) = 0$ for all $n \geq 1$. Define the partial sum process by $S_0 = 0$ and $S_n = X_1 + X_2 + \cdots + X_n$ for $n \geq 1$. Show that $\{S_n\}$ is a martingale.

b. Suppose now the $\{X_n\}$ are independent random variables with $\mathrm{E}(X_n) = 0$ and $\mathrm{V}(X_n) = \sigma^2$ for all $n \geq 1$. Define $M_0 = 0$ and $M_n = S_n^2 - n\sigma^2$ for $n \geq 1$, where $S_n = X_1 + X_2 + \cdots + X_n$ for $n \geq 1$. Prove that $\{M_n\}$ is a martingale.

c. Finally, consider independent random variables $\{X_n\}$ such that $X_n > 0$ and $E(X_n) = 1$ for all $n \geq 1$. Let $M_0 = 1$ and set $M_n = X_1 X_2 \cdots X_n$ for $n \geq 1$. Show that $\{M_n\}$ is a martingale.

Problem 3 (The Probability Generating Function)**.** The *probability generating function (PGF)* of a discrete random variable $X \geq 0$ is defined as

$$G_X(z) = \mathrm{E}(z^X) = \sum_{k=0}^{\infty} z^k \, p_X(k),$$

where z is a complex number with $|z| \leq R$, R denotes the radius of convergence, and $p_X(x)$ denotes the probability mass function.

a. A discrete random variable X is said to have the Poisson distribution with parameter λ if it has a probability mass function given by

$$p_X(k) = \mathrm{P}(X = k) = \frac{\lambda^k e^{-\lambda}}{k!},$$

and we write $X \sim$ Poisson(λ). Find $G_X(z)$.

b. Compute $G_X'(1)$ and $G_X''(1)$, and show that $\mathrm{E}(X) = \mathrm{V}(X) = \lambda$.

If X and Y are two independent non-negative integer-valued random variables, then we have that

$$G_{X+Y}(z) = G_X G_Y.$$

c. Show that if $X_1 \sim$ Poisson(λ_1) and $X_2 \sim$ Poisson(λ_2) are independent, then $X_1 + X_2 \sim$ Poisson$(\lambda_1 + \lambda_2)$.

d. Show that in general if $\{X_n\}$ are independent and $X_i \sim$ Poisson(λ_i) for $i = 1, \ldots, n$, then $X_1 + \ldots + X_n$ is also distributed Poisson with parameter $\lambda_1 + \ldots + \lambda_n$.

Problem 4 (The Characteristic Function)**.** If X is a real-valued random variable with density function $f_X(x)$, the *characteristic function* of X is defined as

$$\phi_X(u) = \mathrm{E}(e^{iuX}) = \int_{-\infty}^{\infty} e^{iux} f_X(x) dx.$$

a. If $X \sim \mathcal{N}(\mu, \sigma^2)$, find $\phi_X(u)$.

If X and Y are two independent real-valued random variables, we also have that

$$\phi_{X+Y}(u) = \phi_X(u)\phi_Y(u).$$

b. Let $X \sim \mathcal{N}(\mu_X, \sigma_X^2)$ and $Z \sim \mathcal{N}(\mu_Z, \sigma_Z^2)$ two independent random variables, and let $Y = X + Z$.

   i. Show $Y \sim \mathcal{N}(\mu_Y, \sigma_Y^2)$, where $\mu_Y = \mu_X + \mu_Z$ and $\sigma_Y^2 = \sigma_X^2 + \sigma_Z^2$.
   ii. Compute $\mathrm{Cov}(Y, X)$.
   iii. Show that $X + Y$ is also normally distributed.

Problem 5 (Stock Prices are Lognormal)**.** Consider a probability space $(\Omega, \mathcal{F}, \mathrm{P})$ and a Brownian motion $\{B_t\}$ generating a filtration $\{\mathcal{F}_t\}_{t\geq 0}$. Consider a non-dividend paying stock whose price S follows a geometric Browninan motion

$$\frac{\mathrm{d}S}{S} = \mu\,\mathrm{d}t + \sigma\,\mathrm{d}B,$$

where μ and σ are constants.

a. Let $X = \ln(S)$. Compute the process followed by $\{X_t\}$ as a function of μ, σ and $\{B_t\}$.
b. Show that
$$S_t = S_0 \exp\left(\left(\mu - \tfrac{1}{2}\sigma^2\right)t + \sigma B_t\right).$$
c. Deduce that $\ln(S_t) \sim \mathcal{N}\left(\ln(S_0) + \left(\mu - \tfrac{1}{2}\sigma^2\right)t, \sigma^2 t\right)$.

Problem 6 (Ito's Lemma)**.** In the following, assume that r, μ and σ are constants.

a. Suppose that the stock price follows a geometric Brownian motion (GBM) with drift μ and instantaneous volatility σ, i.e.,

$$\mathrm{d}S = \mu S\,\mathrm{d}t + \sigma S\,\mathrm{d}B.$$

Show that $Y = 1/S$ also follow a GBM and determine the drift and volatility as a function of μ, and σ.

b. Assume that

$$\mathrm{d}S = rS\,\mathrm{d}t + \sigma S\,\mathrm{d}B.$$

Derive the process followed by the futures price $F(T) = Se^{rT}$ where we interpret T as time-to-maturity.

c. A process S_t is a martingale if $\mathrm{E}(S_T \mid \mathcal{F}_t) = S_t$ for $t < T$. Show that

$$\mathrm{d}S = S\sigma\,\mathrm{d}B$$

is a martingale.

Problem 7 (Multivariate Ito's Lemma)**.**

a. Consider the product $H = XY$ where X and Y are Itô diffusions. Derive the stochastic differential equation for H and verify that

$$\frac{\mathrm{d}H}{H} = \frac{\mathrm{d}X}{X} + \frac{\mathrm{d}Y}{Y} + \left(\frac{\mathrm{d}X}{X}\right)\left(\frac{\mathrm{d}Y}{Y}\right).$$

Under which conditions $\left(\frac{\mathrm{d}X}{X}\right)\left(\frac{\mathrm{d}Y}{Y}\right) = 0$?

b. Suppose a process β satisfies

$$\frac{\mathrm{d}\beta}{\beta} = r\,\mathrm{d}t,$$

and a process Λ satisfies

$$\frac{\mathrm{d}\Lambda}{\Lambda} = -r\,\mathrm{d}t - \lambda\,\mathrm{d}B.$$

Define $\xi = \Lambda\beta$ and show that ξ is a martingale by proving it has zero drift.

c. Consider a stock price process satisfying

$$\frac{\mathrm{d}S}{S} = \mu\,\mathrm{d}t + \sigma\,\mathrm{d}B,$$

and a deflator process satisfying

$$\frac{\mathrm{d}\Lambda}{\Lambda} = -r\,\mathrm{d}t - \lambda\,\mathrm{d}B.$$

Find the value of λ such that the deflated stock price $\xi^S = \Lambda S$ is a martingale.

Problem 8 (The Stochastic Discount Factor)**.** Consider an economy with two tradable assets. The first is a risk-free money-market account β satisfying

$$\frac{\mathrm{d}\beta}{\beta} = r\,\mathrm{d}t,$$

where $\beta_0 = 1$, and r denotes the risk-free rate. The second is a non-dividend paying stock S following geometric Brownian motion:

$$\frac{\mathrm{d}S}{S} = \mu_S\,\mathrm{d}t + \sigma_S\,\mathrm{d}B,$$

where B is a standard Brownian motion under the physical measure $\mathbb{P}$.

Under the no-arbitrage condition, there exists a strictly positive stochastic discount factor (SDF) Λ such that any asset price, when discounted by Λ, becomes a martingale. The SDF satisfies

$$\frac{\mathrm{d}\Lambda}{\Lambda} = -r\,\mathrm{d}t - \lambda\,\mathrm{d}B,$$

where λ is the market price of risk. Throughout, we assume standard integrability conditions ensuring that Itô processes with zero drift are martingales with respect to the filtration $\{\mathcal{F}_t\}$.

a. Let Y to be any traded asset such as the stock or a derivative written on the stock, such that

$$\frac{\mathrm{d}Y}{Y} = \mu_Y\,\mathrm{d}t + \sigma_Y\,\mathrm{d}B.$$

Find λ such that ΛY is a martingale. Explain the intuition of the result.

Consider now a European call $C(S,t)$ with strike K and maturity T.

b. Compute μ_C and σ_C in

$$\frac{\mathrm{d}C}{C} = \mu_C\,\mathrm{d}t + \sigma_C\,\mathrm{d}B.$$

From question (a), we know that

$$\lambda = \frac{\mu_S - r}{\sigma_S} = \frac{\mu_C - r}{\sigma_C}.$$

c. Derive the Black–Scholes PDE

$$\frac{1}{2}\sigma_S^2 S^2 \frac{\partial^2 C}{\partial S^2} + rS\frac{\partial C}{\partial S} + \frac{\partial C}{\partial t} - rC = 0,$$

with terminal condition $C(S,T) = \max(S - K, 0)$.

Problem 9 (The Risk-Neutral Measure)**.** Consider a stochastic discount factor

$$\frac{\mathrm{d}\Lambda}{\Lambda} = -r\,\mathrm{d}t - \lambda\,\mathrm{d}B,$$

where λ is the market price of risk, and define B^* by

$$\mathrm{d}B^* = \mathrm{d}B + \lambda\,\mathrm{d}t.$$

Since $\xi = \Lambda\beta$ is a strictly positive $\mathbb{P}$-martingale, Girsanov's theorem implies that B^* is a standard Brownian motion under the risk-neutral measure $\mathbb{P}^*$, where

$$\frac{\mathrm{d}\mathbb{P}^*}{\mathrm{d}\mathbb{P}} = \xi_T.$$

a. Show that for any traded asset Y, the discounted price process Y/β is a martingale under the risk-neutral measure $\mathbb{P}^*$.

b. Under the risk-neutral measure $\mathbb{P}^*$, derive the stochastic processes governing the stock price S and the call option price C.

c. Show that a long forward contract with delivery price K and expiring at T satisfies the Black-Scholes differential equation. What is the boundary condition in this case?

d. Show that a zero-coupon bond with face value 1 and expiring at T satisfies the Black-Scholes differential equation. What is the boundary condition in this case?