Stochastic Foundations for Finance — Prof. Lorenzo Naranjo

FIN 5380 — Fall 2025

Problem Set 4

Instructions: This problem set is due on 11/24 at 11:59 pm CST and is an individual assignment. All problems must be handwritten. Write full sentences to answer questions that require an explanation. Scan your work and submit a PDF file.

Problem 1. Consider an *arithmetic* Brownian motion process for $\{x\}$ such that

$$\mathrm{d}x = \mu_x \, \mathrm{d}t + \sigma_x \, \mathrm{d}B_x,$$

where μ_x and σ_x are constants.

a. Show that

$$x_t = x_0 + \mu_x t + \sigma_x B_x(t).$$

b. Compute $\mathrm{E}(x_t)$ and $\mathrm{V}(x_t)$.

c. Explain why $x_t \sim \mathcal{N}(\mathrm{E}(x_t), \mathrm{V}(x_t))$.

Consider an *Ornstein–Uhlenbeck* (or mean-reverting) process for $\{y\}$ such that

$$\mathrm{d}y = \kappa(\theta - y) \, \mathrm{d}t + \sigma_y \, \mathrm{d}B_y.$$

d. Show that

$$y_t = e^{-\kappa t} y_0 + \theta \left(1 - e^{-\kappa t}\right) + \sigma_y e^{-\kappa t} \int_0^t e^{\kappa s} dB_y(s).$$

e. Compute $\mathrm{E}(y_t)$ and $\mathrm{V}(y_t)$.

f. Explain why $y_t \sim \mathcal{N}(\mathrm{E}(y_t), \mathrm{V}(y_t))$.

g. Compute $\mathrm{Cov}(x_t, y_t)$.

h. Compute $\lim_{t\to\infty} \mathrm{E}(y_t)$ and explain how this limit shows that the process $\{y\}$ mean-reverts to θ.

Problem 2. Consider a stock price S that follows a geometric Brownian motion with stochastic volatility. The stock price evolves according to

$$\mathrm{d}S = \mu S\,\mathrm{d}t + \sigma S\,\mathrm{d}B_1,$$

where the volatility σ follows a mean-reverting process given by

$$\mathrm{d}\sigma = -\beta\sigma\,\mathrm{d}t + \delta\,\mathrm{d}B_2.$$

The two Brownian motions B_1 and B_2 are correlated with $(\mathrm{d}B_1)(\mathrm{d}B_2) = \rho\,\mathrm{d}t$.

a. Explain why the model allows σ to be negative even though it represents a volatility process.
b. Use Itô's formula to derive the stochastic differential equation for the variance $v = \sigma^2$.
c. Show that the variance follows a process of the form

$$\mathrm{d}v = \kappa(\theta - v)\,\mathrm{d}t + \xi\sqrt{v}\,\mathrm{d}B_2,$$

   and identify the parameters κ, θ, and ξ in terms of β and δ.

Problem 3. Consider two assets whose price processes are given by

$$\frac{\mathrm{d}\mathbf{S}}{\mathbf{S}} = \boldsymbol{\mu}\,\mathrm{d}t + \boldsymbol{\sigma}\,\mathrm{d}\mathbf{B},$$

where d**B** is a vector of three independent Brownian motions B_1, B_2, and B_3. You know that

$$\boldsymbol{\sigma} = \begin{pmatrix} 0.3 & -0.1 & 0.2 \\ 0.15 & 0.2 & -0.05 \end{pmatrix}.$$

a. Compute the instantaneous correlation between the returns of each asset.
b. Find a Brownian motion $B_4 = a_1 B_1 + a_2 B_2 + a_3 B_3$ whose increments are independent from the instantaneous returns of the two assets.

Problem 4. The total instantaneous return of an asset consists of two components:

$$\frac{\mathrm{d}S + D\,\mathrm{d}t}{S} = \frac{\mathrm{d}S}{S} + \frac{D}{S}\,\mathrm{d}t,$$

where $\mathrm{d}S/S$ represents the capital gains and D/S is the dividend yield, which can be interpreted as the rate at which new shares are created through dividend reinvestment. Let $X(t)$ represent the total number of shares owned at time t, which grows according to:

$$\frac{\mathrm{d}X}{X} = \frac{D}{S}\,\mathrm{d}t.$$

a. Show that

$$X_t = X_0 \exp\left(\int_0^t \frac{D_u}{S_u}\,\mathrm{d}u\right).$$

Now define the dividend-reinvested asset price as:

$$V = XS,$$

where V represents the total value of the investment which is equal to the number of shares X multiplied by the current price per share S.

b. Show that

$$\frac{\mathrm{d}V}{V} = \frac{\mathrm{d}S}{S} + \frac{D}{S}\,\mathrm{d}t.$$