

Problem Set 3

Instructions: This problem set is due on 11/17 at 11:59 pm CST and is an individual assignment. All problems must be handwritten. Write full sentences to answer questions that require an explanation. Scan your work and submit a PDF file.

Problem 1. Let $\{B_t\}$ be a Brownian motion defined on a probability space $(\Omega, \mathcal{F}, P)$ endowed with a filtration $\{\mathcal{F}_t\}_{t \geq 0}$. If $s = 0.7$ and $t = 2$, compute:

- a. $P(B_s \leq -0.5)$.
- b. $P(B_t > 1)$.
- c. $P(|B_t| \leq 1.5)$.
- d. $P(|B_t - B_s| > 2)$.

Problem 2. Suppose we divide the interval $[0, T]$ into n equal parts of length $\Delta t = T/n$ given by the partition

$$0 = t_0 < t_1 < \dots < t_n = T.$$

For a real function f , we define the total variation V of f over $[0, T]$ as

$$V_T(f) = \lim_{n \rightarrow \infty} \sum_{j=0}^{n-1} |f(t_{j+1}) - f(t_j)|.$$

If f is differentiable over $[0, T]$, the mean-value theorem implies that over each interval $[t_j, t_{j+1}]$ for $j = 0, 1, \dots, n-1$ there exists $t_j^* \in [t_j, t_{j+1}]$ such that

$$f'(t_j^*) = \frac{f(t_{j+1}) - f(t_j)}{\Delta t},$$

so that $|f(t_{j+1}) - f(t_j)| = |f'(t_j^*)|\Delta t$. Therefore,

$$V_T(f) = \lim_{n \rightarrow \infty} \sum_{j=0}^{n-1} |f'(t_j^*)|\Delta t = \int_0^T |f'(t)|dt.$$

The quadratic variation of f over $[0, T]$ is defined as

$$[f, f]_T = \lim_{n \rightarrow \infty} \sum_{j=0}^{n-1} (f(t_{j+1}) - f(t_j))^2.$$

If the function f is differentiable over $[0, T]$, we can write

$$[f, f]_T = \lim_{n \rightarrow \infty} \sum_{j=0}^{n-1} (f'(t_j^*) \Delta t)^2 = \lim_{n \rightarrow \infty} \Delta t \int_0^T (f'(t))^2 dt.$$

- Complete the reasoning and show that the quadratic variation of a differentiable function f over $[0, T]$ is zero.
- Explain intuitively that a function with non-zero quadratic variation over $[0, T]$ is nowhere differentiable over the same interval.
- Relate the previous result to what we discussed in class about Brownian motion.

Problem 3. Consider an asset that pays a continuous cash flow $ce^{gt} dt$ from time 0 up to time T . The interest rate is r with continuous compounding

- Compute the value of the asset at time 0.
- Compute the value of the asset at time $t < T$.
- What should be the value of the asset at time T ?

Problem 4. Let S be the price of Krypto stock (Ticker: KPTO) that follows a geometric Brownian motion such that

$$dS = \mu S dt + \sigma S dB,$$

where μ and σ are constants. Your sales team is very aggressive and would like to launch a new product called KPTO Quadro that tracks the price of KPTO to the power 4. In other words, the value of this instrument is given by $Y = S^4$. What is the process followed by Y ?

Problem 5. Suppose that the stock price follows a geometric Brownian motion (GBM) with drift μ and instantaneous volatility σ . Show that $Y = Se^{-\mu t}$ also follows a GBM and determine the drift and volatility as a function of μ and σ .

Problem 6. Suppose that the stock price follows a geometric Brownian motion (GBM) with drift r and instantaneous volatility σ , where r is the risk-free rate. Consider the futures price of S at time t and expiring at T , given by $F = Se^{r(T-t)}$. Show that F has zero drift and hence is a martingale.

Normal Distribution Table

The table below computes $\Phi(z) = P(Z \leq z)$ if $Z \sim \mathcal{N}(0, 1)$ and $z \geq 0$. The rows denote the first decimal whereas the columns denote the second decimal. If you need $\Phi(-z)$, you can always use $\Phi(-z) = 1 - \Phi(z)$.

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990