

Problem Set 2

Instructions: This problem set is due on 11/7 at 11:59 pm CST and is an individual assignment. All problems must be handwritten. Write full sentences to answer questions that require an explanation. Scan your work and submit a PDF file.

Problem 1. Let $Y = e^{\mu + \sigma Z}$ where $\mu = 1$, $\sigma = 2$ and $Z \sim \mathcal{N}(0, 1)$. Compute:

- a. $E(Y^{0.5})$
- b. $E(Y^{-1})$

Problem 2. Consider a stock whose price at time T is given by S_T such that,

$$\ln(S_T) \sim \mathcal{N}(\ln(S_0) + (\mu - 0.5\sigma^2)T, \sigma^2 T).$$

The expected return μ is 12% per year and the volatility σ is 35% per year. The current stock price is \$100.

- a. Compute the expected price in 2 years from now.
- b. Compute the mean and standard deviation of the log-spot price in 2 years from now.
- c. Compute the probability that the spot price is less than \$100 in 2 years from now.
- d. Compute the probability that the spot price is greater than \$120 in 2 years from now.

Problem 3. Consider a stock whose price at time T is given by S_T such that,

$$\ln(S_T) \sim \mathcal{N}(\ln(S_0) + (\mu - 0.5\sigma^2)T, \sigma^2 T).$$

The expected return μ is 18% per year and the volatility σ is 32% per year. The current stock price is \$60.

- a. Compute the expected price 9 months from now.
- b. Compute the mean and standard deviation of the log-spot price 9 months from now.

- c. Compute the two-sided 95% confidence interval of $\ln(S_T)$ 9-months from now, and report the corresponding values for S_T .

Problem 4. Consider the biased random walk defined by $S_n = S_0 + \sum_{i=1}^n X_i$, where $P(X_n = 1) = p$ and $P(X_n = -1) = 1 - p = q$ where $p \neq q$. Show that $M_n = (q/p)^{S_n}$ is a martingale.

Problem 5. A discrete random variable X is said to have the Poisson distribution with parameter λ if it has a probability mass function given by

$$p_X(k) = P(X = k) = \frac{\lambda^k e^{-\lambda}}{k!},$$

and we write $X \sim \text{Poisson}(\lambda)$.

A discrete time Poisson process is a sequence of random variables that counts arrivals or events occurring over discrete time steps, where the number of events in each time interval follows a Poisson distribution. Let N_n denote the cumulative count of events up to time $n \geq 0$. The Poisson process is defined so that:

- i. $N_0 = 0$ (no events at the start).
 - ii. The increments $N_n - N_{n-1}$ are independent.
 - iii. Each increment $N_n - N_{n-1} \sim \text{Poisson}(\lambda)$ for some rate parameter $\lambda > 0$.
- a. Show that the process $\{N_n\}$ itself satisfies $N_n \sim \text{Poisson}(n\lambda)$, so the total count grows over time with mean $n\lambda$ and variance $n\lambda$.
 - b. Explain why in each time step, the number of new arrivals is independent of the past, and follows a Poisson distribution with parameter λ .
 - c. Show that $M_n = N_n - n\lambda$ is a martingale.

Normal Distribution Table

The table below computes $\Phi(z) = P(Z \leq z)$ if $Z \sim \mathcal{N}(0, 1)$ and $z \geq 0$. The rows denote the first decimal whereas the columns denote the second decimal. If you need $\Phi(-z)$, you can always use $\Phi(-z) = 1 - \Phi(z)$.

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990