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Stochastic Calculus

Introduction

Let $(\Omega, \mathcal{F}, \mathrm{P})$ be a probability space. Remember that Ω is the set of all the possible outcomes and $\mathcal{F}$ contains all the events $A \subset \Omega$ that we can assert if they happen or not.

In continuous time we define a stochastic process $X_t(\omega)$ as a collection of random variables such that, given an outcome $\omega \in \Omega$, we can determine the path of the stochastic process over time. We can also think about the stochastic process in the opposite way. That is, for a given time $t \leq T$, how does the random variable $X_t(\omega)$ behaves.

A filtration $\{F_t\}$ determines how information is disseminated as we observe a stochastic process. At the very least, we want the filtration to remember what has happened before so that $\mathcal{F}_s \subset \mathcal{F}_t$ when $0 \leq s < t$.

Itô Processes

An Itô process $\{X_t\}$ is a continuous-time stochastic process that can be written as the sum of an ordinary (pathwise) Lebesgue time integral and an Itô stochastic integral:

$$X_t(\omega) = \int_0^t a(s, \omega)\, \mathrm{d}s + \int_0^t b(s, \omega)\, \mathrm{d}B_s(\omega), \tag{1}$$

where the coefficient functions $a(t, \omega)$ and $b(t, \omega)$ are $\mathcal{F}_t$-adapted processes such that $\int_0^t |a(s, \omega)|\, \mathrm{d}s < \infty$, and $\int_0^t b(s, \omega)^2\, \mathrm{d}s < \infty$ almost surely. For compact notation we commonly write

$$\mathrm{d}X = a\, \mathrm{d}t + b\, \mathrm{d}B,$$

with the understanding that this notation represents the integral representation in (**??**).

The stochastic integral is constructed by approximating the integrand with step processes on partitions. For a partition $\Pi = \{t_0, t_1, \dots, t_n\}$ of $[0, T]$ with

$$0 = t_0 < t_1 < \cdots < t_n = t,$$

and $\|\Pi\| \to 0$ as $n \to \infty$, the Itô integral is defined as the mean-square (L^2) limit of Riemann-type sums:

$$I_t(\omega) = \int_0^t b(s,\omega)\,\mathrm{d}B_s(\omega) = \lim_{n\to\infty} \sum_{j=0}^{n-1} b(t_j,\omega)\Delta B_{t_j}(\omega), \tag{2}$$

where the limit is taken in L^2 and, in particular, requires

$$\mathsf{E}(I_t^2) < \infty.$$

The stochastic integral $I_t(\omega)$ is a random process: its value at each time t depends on the sample point ω. Under the usual measurability and square-integrability conditions on the integrand $b(t,\omega)$, the integral admits a modification that is continuous in t for almost every ω (i.e., there exists an indistinguishable version with continuous sample paths). Hence, without loss of generality, we may take I_t to have continuous sample paths.

Itô Integrals are Martingales

Consider first a simple, adapted integrand of the form

$$b(t,\omega) = \sum_{j=0}^{n-1} b(t_j,\omega)\mathbf{1}_{(t_j,t_{j+1}]}(t),$$

so that the Itô integral on this partition is

$$I_t = \sum_{j=0}^{n-1} b(t_j,\omega)\left(B_{t_{j+1}} - B_{t_j}\right).$$

Fix $0 \leq s \leq t$ and let k be the index with $t_k \leq s < t_{k+1}$. Split the sum into the contributions up to time s and those after s so that

$$I_t = \underbrace{\sum_{j=0}^{k-1} b(t_j)\left(B_{t_{j+1}} - B_{t_j}\right)}_{=I_s} + \sum_{j=k}^{n-1} b(t_j)\left(B_{t_{j+1}} - B_{t_j}\right).$$

The first term is $\mathcal{F}_s$-measurable. For each $j \geq k$, $b(t_j)$ is measurable with respect to $\mathcal{F}_{t_j}$ and the increment $B_{t_{j+1}} - B_{t_j}$ is independent of $\mathcal{F}_{t_j}$ and has mean zero, so

$$\mathsf{E}\left[b(t_j)(B_{t_{j+1}} - B_{t_j}) \mid \mathcal{F}_s\right] = 0.$$

Taking conditional expectation yields $\mathsf{E}(I_t \mid \mathcal{F}_s) = I_s$, so I_t is a martingale.

The result for general square-integrable adapted integrands follows by approximating arbitrary predictable integrands by simple ones and using the Itô isometry to pass to the limit.

Conversely, for the Brownian filtration there is the martingale representation theorem: any $\mathcal{F}_t$-adapted martingale $\{M_t\}$ with $\mathsf{E}(M_t^2) < \infty$ for all t can be written as

$$M_t = M_0 + \int_0^t \varphi(s, \omega)\, \mathrm{d}B_s,$$

for a predictable process φ satisfying $\mathsf{E}\left(\int_0^t \varphi(s, \omega)^2\, \mathrm{d}s\right) < \infty$. This gives the converse representation as a stochastic integral with respect to Brownian motion.

Itô Isometry

The Itô isometry expresses the second moment of the stochastic integral:

$$\mathsf{E}(I_t^2) = \mathsf{E}\left(\int_0^t b(s, \omega)^2\, ds\right),$$

i.e., the mean square of the integral equals the expectation of the time-integral of the squared integrand.

Consider a simple, adapted integrand

$$b(s,\omega) = \sum_{j=0}^{n-1} b(t_j,\omega)\mathbf{1}_{(t_j,t_{j+1}]}(s),$$

for which

$$I_t = \sum_{j=0}^{n-1} b(t_j)\left(B_{t_{j+1}} - B_{t_j}\right).$$

Using independence and zero mean of non-overlapping Brownian increments and orthogonality of cross-terms,

$$\begin{aligned}\mathrm{E}(I_t^2) &= \mathrm{E}\left(\sum_{j=0}^{n-1} b(t_j)^2(B_{t_{j+1}} - B_{t_j})^2\right) = \mathrm{E}\left(\sum_{j=0}^{n-1} b(t_j)^2(t_{j+1} - t_j)\right)\\ &= \mathrm{E}\left(\int_0^t b(s,\omega)^2\,\mathrm{d}s\right).\end{aligned}$$

The general result follows by approximating a square-integrable predictable integrand by such simple processes and passing to the limit in L^2. The isometry therefore gives an isometric linear map from the space of square-integrable predictable integrands (with norm given by $\mathrm{E}\int_0^t b(s)^2\,\mathrm{d}s$) into L^2 of the resulting stochastic integrals.

In particular, this yields the square-integrability requirement

$$\mathrm{E}(I_t^2) = \int_0^t b_s^2 ds < \infty.$$

Quadratic Variation

The quadratic variation of a continuous semimartingale X is the (pathwise) limit of squared increments along a refining sequence of partitions $\Pi = \{0 = t_0 < t_1 < \cdots < t_n = t\}$:

$$[X,X]_t = \lim_{n\to\infty} \sum_{j=0}^{n-1} \left(\Delta X_{t_j}\right)^2,$$

whenever the limit exists in probability (or almost surely for continuous local martingales).

Quadratic Variation of the Stochastic Integral

For the Itô integral

$$I_t = \int_0^t b(s,\omega)\,\mathrm{d}B_s,$$

one has the explicit expression for its quadratic variation:

$$[I,I]_t = \int_0^t b^2(s,\omega)\,\mathrm{d}s.$$

We usually summarize this infinitesimally as

$$d[I,I]_t = (dI)^2 = b^2\,\mathrm{d}t.$$

Sketch of proof. Take a simple adapted integrand

$$b(s,\omega) = \sum_j b(t_j,\omega)\mathbf{1}_{(t_j,t_{j+1}]}(s),$$

so that

$$I_t = \sum_{j=0}^{n-1} b(t_j)\Delta B_{t_j}.$$

In this case the quadratic variation along the partition is

$$[I,I]_t = \lim_{n\to\infty} \sum_{j=0}^{n-1} b^2(t_j)(\Delta B_{t_j})^2.$$

To show this limit equals $\int_0^t b^2(s,\omega)\,ds$ in mean square, consider the mean-square difference

$$
\begin{aligned}
\mathsf{E}\left[\left(\sum_{j=0}^{n-1} b(t_j)^2(\Delta B_{t_j})^2 - \sum_{j=0}^{n-1} b(t_j)^2 \Delta t_j\right)^2\right] &= \mathsf{E}\left[\left(\sum_{j=0}^{n-1} b(t_j)^2((\Delta B_{t_j})^2 - \Delta t_j)\right)^2\right] \\
&= \mathsf{E}\left[\sum_{j=0}^{n-1}\sum_{k=0}^{n-1} b(t_j)^2 b(t_k)^2((\Delta B_{t_j})^2 - \Delta t_j)((\Delta B_{t_k})^2 - \Delta t_k)\right] \\
&= \sum_{j=0}^{n-1}\sum_{k=0}^{n-1} \mathsf{E}[b(t_j)^2 b(t_k)^2((\Delta B_{t_j})^2 - \Delta t_j)((\Delta B_{t_k})^2 - \Delta t_k)].
\end{aligned}
$$

If $j < k$ the factors involving disjoint increments are independent, so the cross-terms vanish:

$$
\mathsf{E}\left[b(t_j)^2 b(t_k)^2((\Delta B_{t_j})^2 - \Delta t_j)((\Delta B_{t_k})^2 - \Delta t_k)\right] = 0.
$$

Using $\Delta B_{t_j} \sim \mathcal{N}(0, \Delta t_j)$ we have $\mathsf{E}[(\Delta B_{t_j})^2] = \Delta t_j$ and $\mathsf{V}[(\Delta B_{t_j})^2] = 2(\Delta t_j)^2$. Hence

$$
\begin{aligned}
\mathsf{E}\left[\left(\sum_{j=0}^{n-1} b(t_j)^2(\Delta B_{t_j})^2 - \sum_{j=0}^{n-1} b(t_j)^2 \Delta t_j\right)^2\right] &= \sum_{j=0}^{n-1} \mathsf{E}[b(t_j)^4((\Delta B_{t_j})^2 - \Delta t_j)^2] \\
&= \sum_{j=0}^{n-1} \mathsf{E}[b(t_j)^4]\,\mathsf{E}[((\Delta B_{t_j})^2 - \Delta t_j)^2] \\
&= \sum_{j=0}^{n-1} \mathsf{E}[b(t_j)^4] 2(\Delta t_j)^2 \\
&\leq \|\Pi\|^2 \sum_{j=0}^{n-1} \mathsf{E}[b^4(t_j)](\Delta t_j).
\end{aligned}
$$

Under the integrability assumption $\int_0^t \mathsf{E}\left[b^4(s)\right] \mathrm{d}s = \mathsf{E}\left(\int_0^t b^4(s)\, \mathrm{d}s\right) < \infty$, the right-hand side tends to zero as $\|\Pi\| \to 0$. Therefore

$$
\sum_{j=0}^{n-1} b(t_j)^2(\Delta B_{t_j})^2 \xrightarrow[L^2]{} \int_0^t b^2(s, \omega)\, \mathrm{d}s,
$$

which yields the desired quadratic variation identity. □

Quadratic Variation of the Itô Process

For the general Itô process defined in (**??**), the quadratic variation is equivalent to the stochastic integral, expressed as:

$$[X,X]_t \;=\; \int_0^t b(s,\omega)^2 \,\mathrm{d}s.$$

The drift term $\int_0^t a(s,\omega)\,\mathrm{d}s$ contributes zero to the quadratic variation because it is of bounded variation. Specifically, its increments are of order Δt, leading to their squares being of order $(\Delta t)^2$, which vanish in the limit. Thus, only the stochastic integral contributes to the quadratic variation.

To illustrate this, consider simple adapted integrands:

$$a(s,\omega) = \sum_j a(t_j,\omega)\mathbf{1}_{(t_j,t_{j+1}]}(s), \qquad b(s,\omega) = \sum_j b(t_j,\omega)\mathbf{1}_{(t_j,t_{j+1}]}(s).$$

The increment of X over the interval $[t_j, t_{j+1}]$ is given by:

$$\Delta X_{t_j}(\omega) = a(t_j,\omega)\Delta t_j + b(t_j,\omega)\Delta B_{t_j}(\omega).$$

Consequently, we have:

$$\left(\Delta X_{t_j}\right)^2 = a(t_j)^2(\Delta t_j)^2 + 2a(t_j)b(t_j)(\Delta t_j)(\Delta B_{t_j}) + b(t_j)^2(\Delta B_{t_j})^2.$$

In the limit as $n \to \infty$, we analyze the expression:

$$\lim_{n\to\infty} \sum_{j=0}^{n-1} \left(\Delta X_{t_j}\right)^2.$$

The first term vanishes because:

$$\sum_{j=0}^{n-1} a(t_j,\omega)^2(\Delta t_j)^2 \leq \|\Pi\| \int_0^t a(s,\omega)^2\,\mathrm{d}s \to 0 \quad \text{as } n\to\infty,$$

assuming that $\int_0^t a(s,\omega)^2\,\mathrm{d}s < \infty$ almost surely.

Next, we analyze the second term by considering the sum:

$$S_n = \sum_{j=0}^{n-1} 2a(t_j,\omega)b(t_j,\omega)\Delta t_j \Delta B_{t_j}.$$

We can express the expected value as:

$$\mathsf{E}(S_n^2) = \mathsf{E}\left[\left(\sum_{j=0}^{n-1} 2a(t_j)b(t_j)\Delta t_j \Delta B_{t_j}\right)^2\right] = \mathsf{E}\left[\sum_{j=0}^{n-1}\sum_{k=0}^{n-1} 4a(t_j)b(t_j)a(t_k)b(t_k)\Delta t_j \Delta t_k \Delta B_{t_j}\Delta B_{t_k}\right].$$

Since for $j < k$, the terms $a(t_j)b(t_j)a(t_k)b(t_k)\Delta t_j \Delta t_k \Delta B_{t_j}$ are independent of ΔB_{t_k}, all cross-terms vanish, leading to:

$$\begin{aligned}
\mathsf{E}(S_n^2) &= \mathsf{E}\left[\sum_{j=0}^{n-1} 4a^2(t_j)b^2(t_j)(\Delta t_j)^2(\Delta B_{t_j})^2\right] \\
&= \sum_{j=0}^{n-1} \mathsf{E}\left[4a^2(t_j)b^2(t_j)(\Delta t_j)^2\right]\mathsf{E}\left[(\Delta B_{t_j})^2\right] \\
&= \sum_{j=0}^{n-1} \mathsf{E}\left[4a^2(t_j)b^2(t_j)(\Delta t_j)^3\right] \\
&\leq \|\Pi\|^2 \sum_{j=0}^{n-1} \mathsf{E}\left[4a^2(t_j)b^2(t_j)\Delta t_j\right].
\end{aligned}$$

As a result, we find:

$$\lim_{n\to\infty} \mathsf{E}(S_n^2) \leq \lim_{n\to\infty} \|\Pi\|^2 \int_0^t \mathsf{E}\left[4a^2(s)b^2(s)\right] ds = 0,$$

which implies that $\lim_{n\to\infty} S_n = 0$ in L^2 (and thus in probability).

Finally, we have already established that:

$$\lim_{n\to\infty} \sum_{j=0}^{n-1} b^2(s,\omega)(\Delta B_{t_j})^2 = \int_0^t b^2(s,\omega)\,\mathrm{d}s$$

in L^2. Therefore, we conclude:

$$\sum_j \left(\Delta X_{t_j}\right)^2 \xrightarrow[L^2]{} \int_0^t b(s,\omega)^2 ds.$$

This result can be extended to general square-integrable adapted coefficients through approximation with simple processes.

Itô's Formula

Itô's formula generalizes the chain rule to stochastic processes. It provides the precise way to compute the differential of a smooth function of an Itô process.

Itô's Formula (General Form)

Let X_t be an Itô process satisfying

$$\mathrm{d}X_t = a(t,\omega)\,\mathrm{d}t + b(t,\omega)\,\mathrm{d}B_t,$$

and let $f(x,t)$ be a twice continuously differentiable function in x and once continuously differentiable in t. Then $Y_t = f(X_t,t)$ is also an Itô process with

$$\mathrm{d}Y_t = \left(a(t,\omega)\frac{\partial f}{\partial x} + \frac{1}{2}b^2(t,\omega)\frac{\partial^2 f}{\partial x^2} + \frac{\partial f}{\partial t}\right)\mathrm{d}t + b(t,\omega)\frac{\partial f}{\partial x}\,\mathrm{d}B_t.$$

Sketch of Proof. We begin with a second-order Taylor expansion of $f(X_{t+\Delta t}, t+\Delta t)$ around (X_t,t):

$$\begin{aligned} f(X_{t+\Delta t}, t+\Delta t) \approx f(X_t,t) &+ \frac{\partial f}{\partial t}\Delta t + \frac{\partial f}{\partial x}\Delta X_t \\ &+ \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(\Delta X_t)^2 + \frac{1}{2}\frac{\partial^2 f}{\partial t^2}(\Delta t)^2 + \frac{\partial^2 f}{\partial x \partial t}\Delta X_t \Delta t + \ldots \end{aligned}$$

Therefore, the increment in f is

$$\Delta f = f(X_{t+\Delta t}, t+\Delta t) - f(X_t,t) \approx \frac{\partial f}{\partial t}\Delta t + \frac{\partial f}{\partial x}\Delta X_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(\Delta X_t)^2 + \text{higher order terms.}$$

For an Itô process with $\mathrm{d}X_t = a(t,\omega)\,\mathrm{d}t + b(t,\omega)\,\mathrm{d}B_t$, the increment over $[t, t + \Delta t]$ is

$$\Delta X_t = a(t,\omega)\Delta t + b(t,\omega)\Delta B_t.$$

We already saw that $(\Delta X_t)^2 \approx b^2(t)\Delta t$. Thus,

$$\Delta f \approx \left(\frac{\partial f}{\partial t} + a(t,\omega)\frac{\partial f}{\partial x} + \frac{1}{2}b^2(t,\omega)\frac{\partial^2 f}{\partial x^2}\right)\Delta t + b(t,\omega)\frac{\partial f}{\partial x}\Delta B_t.$$

Passing to the limit as $\Delta t \to 0$ yields Itô's formula in differential form:

$$\mathrm{d}f = \left(\frac{\partial f}{\partial t} + a(t,\omega)\frac{\partial f}{\partial x} + \frac{1}{2}b^2(t,\omega)\frac{\partial^2 f}{\partial x^2}\right)\mathrm{d}t + b(t,\omega)\frac{\partial f}{\partial x}\,\mathrm{d}B_t.$$

The additional term $\frac{1}{2}b^2\frac{\partial^2 f}{\partial x^2}\,\mathrm{d}t$ arises because of the non-zero quadratic variation $(\mathrm{d}X_t)^2 = b^2\,\mathrm{d}t$. This term has no classical analogue—in ordinary calculus where paths are smooth, the chain rule contains no such correction. The presence of this drift term is the defining feature of stochastic calculus and reflects the fractal, rough nature of Brownian motion. □