# The Normal and Lognormal Distributions

## The Normal Distribution

We say that a real-valued random variable (RV) $X$ is normally distributed with mean $\mu$ and standard deviation $\sigma$ if its *probability density function* (PDF) is:

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

and we usually write $X \sim \mathcal{N}(\mu, \sigma^2)$. The parameters $\mu$ and $\sigma$ are related to the first and second moments of $X$.

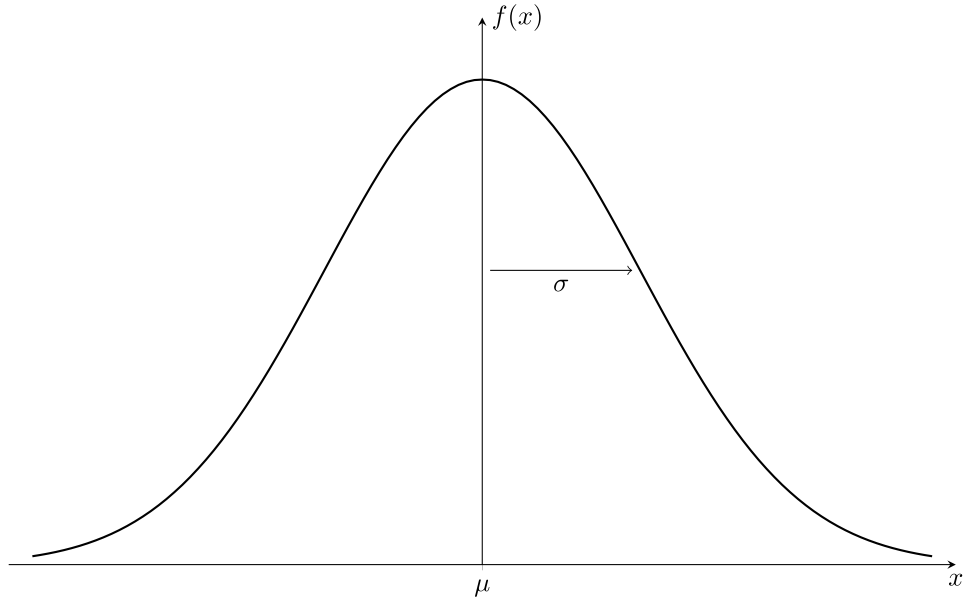



**Figure 1:** The figure shows the density function of a normally distributed random variable with mean $\mu$ and standard deviation $\sigma$.

> Moments of the Normal Distribution
>
> The parameter $\mu$ is the mean or expectation of $X$ while $\sigma$ denote its standard deviation. The variance of $X$ is given by $\sigma^2$.

Proof

Let $X = \mu + \sigma Z$ where $Z \sim \mathcal{N}(0,1)$. Start by defining $f(z) = e^{-\frac{1}{2}z^2}$, which implies that $f'(z) = -ze^{-\frac{1}{2}z^2}$ and $f''(x) = z^2e^{-\frac{1}{2}z^2} - e^{-\frac{1}{2}z^2}$. We can then write:

$$\begin{aligned} ze^{-\frac{1}{2}z^2} &= -f'(z) \\ z^2e^{-\frac{1}{2}z^2} &= f''(x) + f(z) \end{aligned}$$

Then,

$$\begin{aligned} \mathsf{E}(Z) &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} ze^{-\frac{1}{2}z^2}\,dz \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} -f'(z)\,dz \\ &= \frac{1}{\sqrt{2\pi}} \left( -f(z)\Big|_{-\infty}^{\infty} \right) \\ &= 0, \\ \mathsf{E}(Z^2) &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} z^2e^{-\frac{1}{2}z^2}\,dz \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f''(x) + f(z)\,dz \\ &= \frac{1}{\sqrt{2\pi}} \left( f'(z)\Big|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} f(z)\,dz \right) \\ &= \frac{1}{\sqrt{2\pi}} (0 + \sqrt{2\pi}) \\ &= 1, \\ \mathsf{V}(Z) &= \mathsf{E}(Z^2) - \mathsf{E}(Z)^2 \\ &= 1. \end{aligned}$$

Note that we used the fact that

$$\int_{-\infty}^{\infty} f(z)\,dz = \sqrt{2\pi}.$$

We can now compute $\mathsf{E}(X) = \mu + \sigma\,\mathsf{E}(Z) = \mu$ and $\mathsf{V}(X) = \sigma^2\,\mathsf{V}(Z) = \sigma^2$. $\square$

As with any real-valued random variable $X$, in order to compute the probability that $X \leq x$ we need to integrate the density function from $-\infty$ to $x$:

$$\mathsf{P}(X \leq x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(u-\mu)^2}{2\sigma^2}} du.$$

The function $F(x) = \mathsf{P}(X \leq x)$ is called the *cumulative distribution function* of $X$. The *Leibniz integral rule* implies that $F'(x) = f(x)$.

### The Standard Normal Distribution

An important case of normally distributed random variables is when $\mu = 0$ and $\sigma = 1$. In this case we say that $Z \sim \mathcal{N}(0,1)$ has the *standard normal distribution* and its cumulative distribution function is usually denoted by the capital Greek letter $\Phi$ (phi), and is defined by the integral:

$$\Phi(z) = \mathsf{P}(Z \leq z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx.$$

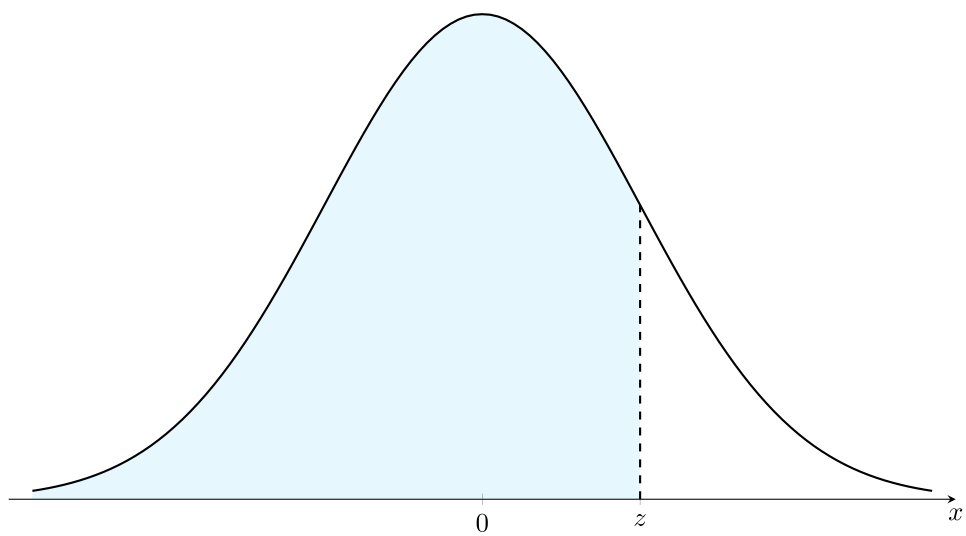



**Figure 2:** The blue shaded area represents $\Phi(z)$.

Since the integral cannot be solved in closed-form, the probability must then be obtained from a table or using a computer. For example, in R we can compute $\Phi(-0.4)$ by typing the following:

```r
pnorm(-0.4)
```

```
[1] 0.3445783
```

If you prefer to use Excel, you need to type in a cell `=norm.s.dist(-0.4,TRUE)`, which yields the same answer.

### Left-Tail Probability

Knowing how to compute or approximate $\Phi(z)$ allows us to compute $\mathrm{P}(X \leq x)$ when $X \sim \mathcal{N}(\mu, \sigma^2)$ since $Z = \frac{X-\mu}{\sigma} \sim \mathcal{N}(0,1)$:

$$\begin{aligned}\mathrm{P}(X \leq x) &= \mathrm{P}\left(\frac{X-\mu}{\sigma} \leq \frac{x-\mu}{\sigma}\right)\\ &= \mathrm{P}\left(Z \leq \frac{x-\mu}{\sigma}\right)\\ &= \Phi\left(\frac{x-\mu}{\sigma}\right)\end{aligned}$$

where $Z = \dfrac{X-\mu}{\sigma} \sim \mathcal{N}(0,1)$ is called a Z-score.

**Example 1.** Suppose that $X \sim \mathcal{N}(\mu, \sigma^2)$ with $\mu = 10$ and $\sigma = 25$. What is the probability that $X \leq 0$?

$$\begin{aligned}\mathrm{P}(X \leq 0) &= \mathrm{P}\left(Z \leq \tfrac{0-10}{25}\right)\\ &= \Phi(-0.40)\\ &= 0.3446.\end{aligned}$$

□

### Right-Tail Probability

For a random variable $X$, the *right-tail* probability is defined as $\mathrm{P}(X > x)$. Since $\mathrm{P}(X \leq x) + \mathrm{P}(X > x) = 1$, we have that:

$$\mathrm{P}(X > x) = 1 - \mathrm{P}(X \leq x).$$

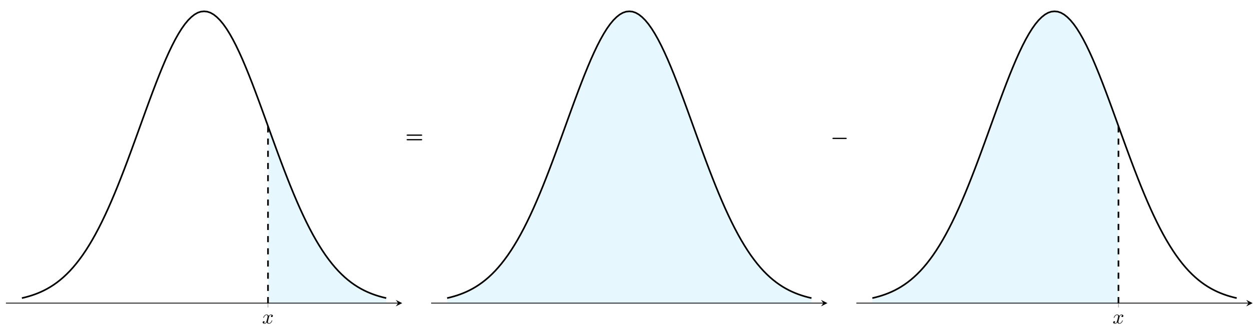



**Figure 3:** The right-tail probability is the probability of the whole distribution, which is one, minus the left-tail probability.

**Example 2.** Suppose that $X \sim \mathcal{N}(\mu, \sigma^2)$ with $\mu = 10$ and $\sigma = 25$. What is the probability that $X > 12$?

$$\begin{aligned} \mathrm{P}(X \leq 12) &= \mathrm{P}\left(Z \leq \tfrac{12-10}{25}\right) \\ &= \Phi(0.08) \\ &= 0.5319. \end{aligned}$$

Therefore, $\mathrm{P}(X > 12) = 1 - 0.5319 = 0.4681$. □

### Interval Probability

The probability that a random variable $X$ falls within an interval $(X_1, X_2]$ is given by $\mathrm{P}(x_1 < X \leq x_2) = \mathrm{P}(X \leq x_2) - \mathrm{P}(X \leq x_1)$.

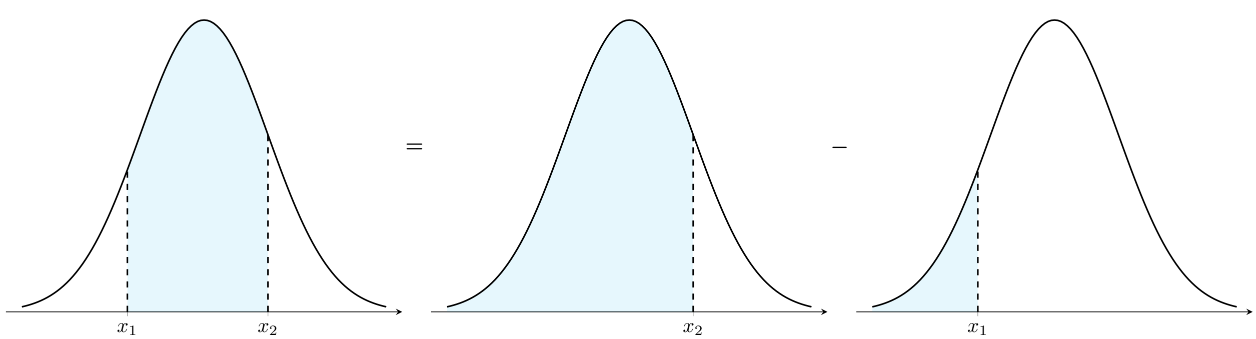



**Figure 4:** If you subtract the area to the left of $x_1$ to the area that is to the left of $x_2$ you obtain the probability of $x_1 < X \leq x_2$.

**Example 3.** Suppose that $X \sim \mathcal{N}(\mu, \sigma^2)$ with $\mu = 10$ and $\sigma = 25$. What is the probability that $2 < X \leq 14$?

$$\begin{aligned} \mathsf{P}(X \leq 14) &= \mathsf{P}\left(Z \leq \tfrac{14-10}{25}\right) \\ &= \Phi(0.16) \\ &= 0.5636, \\ \mathsf{P}(X \leq 2) &= \mathsf{P}\left(Z \leq \tfrac{2-10}{25}\right) \\ &= \Phi(-0.32) \\ &= 0.3745. \end{aligned}$$

Therefore, $\mathsf{P}(2 < X \leq 14) = 0.5636 - 0.3745 = 0.1891$. □

## The Lognormal Distribution

If $X \sim \mathcal{N}(\mu, \sigma^2)$, then $Y = e^X$ is said to be *lognormally distributed* with the same parameters. The pdf of a lognormally distributed random variable $Y$ can be obtained from the pdf of $X$.

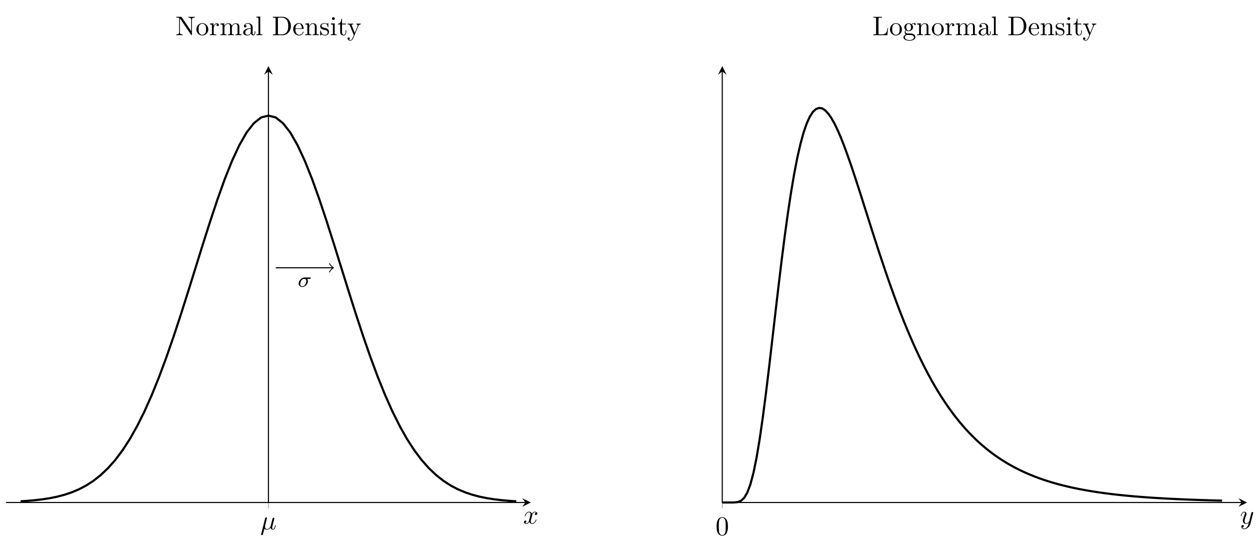



**Figure 5:** The figure shows the difference between a normal and a lognormal PDF with the same parameters.

Lognormal Density

If $Y$ is lognormally distributed with parameters $\mu$ and $\sigma^2$, the PDF of $Y$ is given by:

$$f(y) = \frac{1}{y\sqrt{2\pi\sigma^2}} e^{-\frac{(\ln(y)-\mu)^2}{2\sigma^2}}.$$

Proof

Let $Y = e^X$ where $X = \mu + \sigma Z$ and $Z \sim \mathcal{N}(0,1)$. Then,

$$\begin{aligned} \mathrm{P}(Y \leq y) &= \mathrm{P}(X \leq \ln(y)) \\ &= \int_{-\infty}^{\ln(y)} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}\, dx. \end{aligned}$$

Let's define $z = e^x$. This implies that $x = \ln(z)$, which in turn implies that $dx = (1/z)dz$. Therefore,

$$\mathrm{P}(Y \leq y) = \int_{-\infty}^{y} \frac{1}{z\sqrt{2\pi\sigma^2}} e^{-\frac{(\ln(z)-\mu)^2}{2\sigma^2}}\, dz.$$

Thus, the integrand of the previous expression is the probability density function of $Y$. $\square$

Unlike the normal density, the lognormal density function is not symmetric around its mean. Normally distributed variables can take values in $(-\infty, \infty)$, whereas lognormally distributed variables are always positive.

### Computing Probabilities

We can use the fact that the logarithm of a lognormal random variable is normally distributed to compute cumulative probabilities.

**Example 4.** Let $Y = e^{4+1.5Z}$ where $Z \sim \mathcal{N}(0,1)$. What is the probability that $Y \leq 100$?

$$\begin{aligned}
\mathsf{P}(Y \leq 100) &= \mathsf{P}(e^X \leq 100) \\
&= \mathsf{P}(X \leq \ln(100)) \\
&= \mathsf{P}\left(Z \leq \tfrac{\ln(100)-4}{1.5}\right) \\
&= \Phi(0.4034) \\
&= 0.6567
\end{aligned}$$

Therefore, there is a 65.67% chance that $Y$ is less than or equal 100. □

### Moments

**Moments of a Lognormal Distribution**

Let $Y = e^{\mu+\sigma Z}$ where $Z \sim \mathcal{N}(0,1)$. We have that:

$$\begin{aligned}
\mathsf{E}(Y) &= e^{\mu+0.5\sigma^2} \\
\mathsf{V}(Y) &= e^{2\mu+\sigma^2}(e^{\sigma^2}-1) \\
\mathsf{SD}(Y) &= \mathsf{E}(Y)\sqrt{e^{\sigma^2}-1}
\end{aligned}$$

Proof

$$\begin{aligned}
\mathsf{E}(Y) &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} e^x \, dx \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}+x} \, dx \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-(\mu+\sigma^2))^2}{2\sigma^2}+(\mu+0.5\sigma^2)} \, dx \\
&= e^{\mu+0.5\sigma^2} \underbrace{\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-(\mu+\sigma^2))^2}{2\sigma^2}} \, dx}_{=1} \\
&= e^{\mu+0.5\sigma^2}
\end{aligned}$$

Using the fact that $\alpha X \sim \mathcal{N}(\alpha\mu, (\alpha\sigma)^2)$, it is also possible to compute the expectation of powers of lognormally distributed variables:

$$\mathsf{E}(Y^\alpha) = \mathsf{E}(e^{\alpha X}) = e^{\alpha\mu + 0.5(\alpha\sigma)^2}.$$

This is useful to compute the variance and standard deviation of $Y$:

$$\begin{aligned} \mathsf{V}(Y) &= \mathsf{E}(Y^2) - (\mathsf{E}(Y))^2 \\ &= e^{2\mu+2\sigma^2} - e^{2\mu+\sigma^2} \\ &= e^{2\mu+\sigma^2}(e^{\sigma^2} - 1) \\ \mathsf{SD}(Y) &= \sqrt{\mathsf{V}(Y)} \\ &= \mathsf{E}(Y)\sqrt{e^{\sigma^2} - 1} \end{aligned}$$

□

**Example 5.** Let $Y = e^{4+1.5Z}$ where $Z \sim \mathcal{N}(0,1)$. The expectation and standard deviation of $Y$ are:

$$\mathsf{E}(Y) = e^{4+0.5(1.5^2)} = 168.17$$
$$\mathsf{SD}(Y) = 168.17\sqrt{e^{1.5^2} - 1} = 489.95$$

□

## Appendix

### Percentiles

For a standard normal variable $Z$, a *right-tail percentile* is the value $z_\alpha$ above which we obtain a certain probability $\alpha$. Mathematically, this means finding $z_\alpha$ such that:

$$\mathsf{P}(Z > z_\alpha) = \alpha \Leftrightarrow \mathsf{P}(Z \le z_\alpha) = 1 - \alpha.$$

This implies that $\Phi(z_\alpha) = 1 - \alpha$, or $z_\alpha = \Phi^{-1}(1-\alpha)$, where $\Phi^{-1}(\cdot)$ denotes the inverse function of $\Phi(\cdot)$. Again, there is no closed-form expression for this function and we need a computer to obtain the values. For example, say that $\alpha = 0.025$. In R we could compute $z_\alpha = \Phi^{-1}(0.975)$ by using the function `qnorm` as follows:

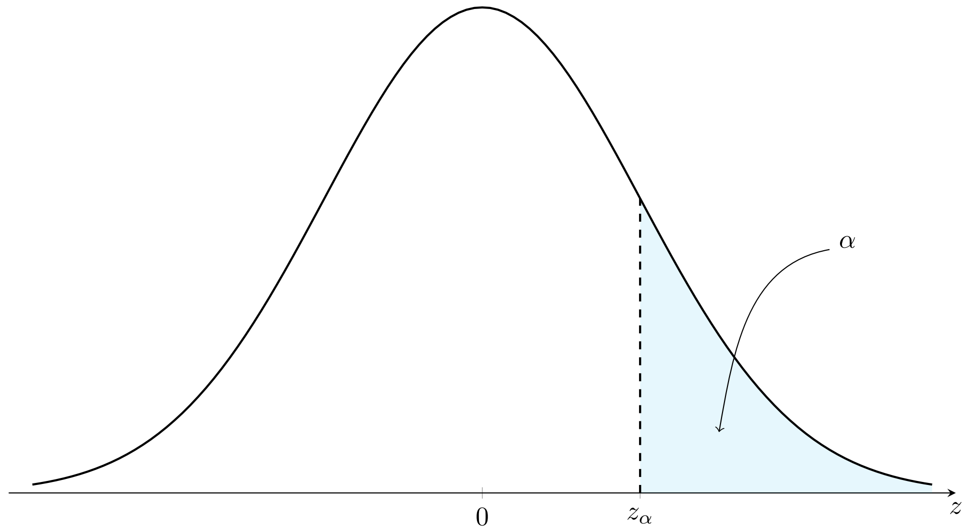



**Figure 6:** The right-tail percentile is the value $z_\alpha$ that gives an area to the right equal to $\alpha$.

```r
qnorm(0.975)
```

```
[1] 1.959964
```

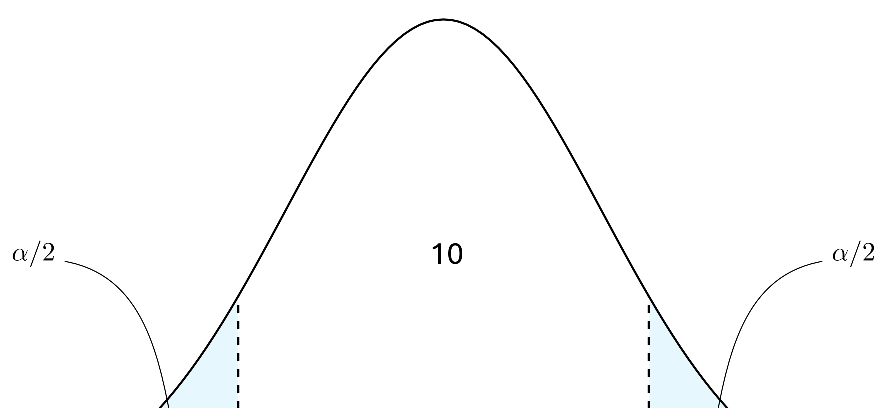
In Excel the function `=norm.s.inv(0.975)` provides the same result.

The following table shows common values for $z_\alpha$:

| $\alpha$ | $z_\alpha$ |
|---|---|
| 0.050 | 1.64 |
| 0.025 | 1.96 |
| 0.010 | 2.33 |
| 0.005 | 2.58 |

A $(1-\alpha)$ *two-sided confidence interval* (CI) defines left and right percentiles such that the probability on each side is $\alpha/2$. For a standard normal variable $Z$, the symmetry of its pdf implies:

$$\mathrm{P}(Z \leq -z_{\alpha/2}) = \mathrm{P}(Z > z_{\alpha/2}) = \alpha/2$$



**Example 6.** Since $z_{2.5\%} = 1.96$, the 95% confidence interval of $Z$ is $[-1.96, 1.96]$. This means that if we randomly sample this variable 100,000 times, approximately 95,000 observations will fall inside this interval. $\square$

If $X \sim \mathcal{N}(\mu, \sigma^2)$, its confidence interval is determined by $\xi$ and $\zeta$ such that:

$$\begin{aligned}
&\mathrm{P}(X \leq \xi) = \alpha/2 \\
&\qquad \Rightarrow \mathrm{P}(Z \leq \tfrac{\xi-\mu}{\sigma}) = \alpha/2, \\
&\mathrm{P}(X > \zeta) = \alpha/2 \\
&\qquad \Rightarrow \mathrm{P}(Z > \tfrac{\zeta-\mu}{\sigma}) = \alpha/2,
\end{aligned}$$

which implies that $-z_{\alpha/2} = \frac{\xi-\mu}{\sigma}$ and $z_{\alpha/2} = \frac{\zeta-\mu}{\sigma}$.The $(1-\alpha)$ confidence interval for $X$ is then $[\mu - z_{\alpha/2}\sigma, \mu + z_{\alpha/2}\sigma]$.

**Example 7.** Suppose that $X \sim \mathcal{N}(\mu, \sigma^2)$ with $\mu = 10$ and $\sigma = 25$. Since $z_{2.5\%} = 1.96$, the 95% confidence interval of $X$ is:

$$[10 - 1.96(25), 10 + 1.96(25)] = [-39, 59].$$

$\square$

We could also apply the same priciple for a lognormal random variable. Let $Y = e^{\mu+\sigma Z}$ where $Z \sim \mathcal{N}(0,1)$. We then have that

$$\begin{aligned}
&-z_{\alpha/2} < Z \leq z_{\alpha/2} \\
&\qquad \Rightarrow \mu - \sigma z_{\alpha/2} < \mu + \sigma Z \leq \mu + \sigma z_{\alpha/2} \\
&\qquad \Rightarrow e^{\mu - \sigma z_{\alpha/2}} < e^{\mu+\sigma Z} \leq e^{\mu+\sigma z_{\alpha/2}}
\end{aligned}$$

The $(1-\alpha)$ confidence interval for $Y$ (centered araound the mean of $\ln(Y)$) is $[e^{\mu-\sigma z_{\alpha/2}}, e^{\mu+\sigma z_{\alpha/2}}]$.

**Example 8.** Let $Y = e^{4+1.5Z}$ where $Z \sim \mathcal{N}(0,1)$. The 95% confidence interval for $Y$ is:

$$[e^{4-1.96(1.5)}, e^{4+1.96(1.5)}] = [2.89, 1032.71].$$

$\square$