Martingales in Discrete Time

All the stochastic processes that we will study in this class will be generated from a sequence of random variables. The specific sequence used to generate the stochastic process $\{M_n : 0 \leq n < \infty\}$ determines the information set $\mathcal{F}_n$ available at time n. As time passes, we learn more about the processes and what was used to generate the process.

We call the collection of information sets $\{\mathcal{F}_n : 0 \leq n < \infty\}$ a filtration. Since information accumulates over time, we always have that $\mathcal{F}_m \subset \mathcal{F}_n$ for $m < n$.

Martingales

Earlier, we defined the simple random walk as the process $\{S_n : 0 \leq n < \infty\}$, built from a sequence $\{X_n : 1 \leq n < \infty\}$ of independent random variables, where

$$S_n = S_0 + \sum_{i=1}^{n} X_i$$

for $n \geq 1$. Each X_i is a fair coin toss, taking values -1 or 1 with probability $1/2$. At time n, the outcomes of the first n coin tosses are known, so the information set is $\mathcal{F}_n = \{X_1, X_2, \dots, X_n\}$, containing all information revealed by $X_1, X_2, \dots, X_n$.

We can then see that

$$\mathsf{E}(S_{n+1} \mid \mathcal{F}_n) = \mathsf{E}(S_n + X_{n+1} \mid \mathcal{F}_n) = S_n + \mathsf{E}(X_{n+1} \mid \mathcal{F}_n).$$

There is an implicit assumption that the coin tosses are independent from each other. Intuitively, we do not expect to gain any information from the first n tosses to determine the next one. Thus, $\mathsf{E}(X_{n+1} \mid \mathcal{F}_n) = \mathsf{E}(X_{n+1} = 0)$, implying

$$\mathsf{E}(S_{n+1} \mid \mathcal{F}_n) = S_n.$$

The process just defined is what we call a *martingale*. A martingale is a stochastic process $\{M_n\}$ adapted to a filtration $\{\mathcal{F}_n\}$ such that for all n,

$$\mathsf{E}(M_{n+1} \mid \mathcal{F}_n) = M_n.$$

In other words, given all the information up to time n, the expected value of the next observation is equal to the current value. This property captures the idea that, on average, the process does not "drift" up or down given the available information.

Example 1. We can generalize the simple random walk by defining $\{X_n : 1 \leq n < \infty\}$ as a sequence of independent random variables such that $\mathsf{E}(X_n) = 0$ for all $n \geq 1$. We do not assume any specific distribution nor we assume that the random variables are identically distributed. We can then define

$$S_n = S_0 + \sum_{i=1}^{n} X_i.$$

We can do as before to find

$$\mathsf{E}(S_{n+1} \mid \mathcal{F}_n) = \mathsf{E}(S_n + X_{n+1} \mid \mathcal{F}_n) = S_n + \mathsf{E}(X_{n+1} \mid \mathcal{F}_n) = S_n + \mathsf{E}(X_{n+1}) = S_n,$$

showing that $\{S_n\}$ is a martingale with respect to $\{\mathcal{F}_n\}$.

Example 2. Let $\{X_n : 1 \leq n < \infty\}$ be a sequence of independent random variables with $\mathsf{E}(X_n) = 0$ and $\mathsf{V}(X_n) = \sigma^2$ for all $n \geq 1$. We do not require the random variables to be identically distributed, but each has mean zero and variance σ^2. Define S_n as in Example 1, and set

$$M_n = S_n^2 - n\sigma^2.$$

Expanding M_{n+1},

$$M_{n+1} = (S_n + X_{n+1})^2 - (n+1)\sigma^2 = S_n^2 + 2S_n X_{n+1} + X_{n+1}^2 - (n+1)\sigma^2.$$

Taking conditional expectation,

$$\mathsf{E}(M_{n+1} \mid \mathcal{F}_n) = S_n^2 + 2S_n\,\mathsf{E}(X_{n+1} \mid \mathcal{F}_n) + \mathsf{E}(X_{n+1}^2 \mid \mathcal{F}_n) - (n+1)\sigma^2.$$

Because X_{n+1} is independent of $\mathcal{F}_n$, we have $\mathsf{E}(X_{n+1} \mid \mathcal{F}_n) = 0$ and $\mathsf{E}(X_{n+1}^2 \mid \mathcal{F}_n) = \sigma^2$.

Therefore,

$$\mathrm{E}(M_{n+1} \mid \mathcal{F}_n) = S_n^2 + \sigma^2 - (n+1)\sigma^2 = S_n^2 - n\sigma^2 = M_n.$$

Thus, $\{M_n\}$ is a martingale with respect to $\{\mathcal{F}_n\}$.

Example 3. Let $\{X_n : 1 \leq n < \infty\}$ be a sequence of independent random variables with $X_n > 0$ and $\mathrm{E}(X_n) = 1$ for all $n \geq 1$. Set $M_0 = 1$ and define $M_n = X_1 X_2 \cdots X_n$. Since $M_{n+1} = M_n X_{n+1}$, we have

$$\mathrm{E}(M_{n+1} \mid \mathcal{F}_n) = M_n \, \mathrm{E}(X_{n+1} \mid \mathcal{F}_n) = M_n \, \mathrm{E}(X_{n+1}) = M_n.$$

Therefore, $\{M_n\}$ is a martingale with respect to $\{\mathcal{F}_n\}$.

Stopped Processes

A random variable τ taking values in $\{0, 1, 2, ...\} \cup \{\infty\}$ is called a stopping time with respect to $\{\mathcal{F}_n : 0 \leq n \leq \infty\}$ if

$$\{\tau \leq n\} \in \mathcal{F}_n \text{ for all} 0 \leq n < \infty.$$

If $\tau < \infty$ almost surely, we can define the stopped process

$$Y_\tau = \sum_{k=0}^{\infty} \mathbf{1}_{\{\tau = k\}} Y_k.$$

In cases where $\mathrm{P}(\tau = \infty) > 0$, we can always define the truncated stopping time $n \wedge \tau = \min\{n, \tau\}$. The stopped process $Y_{n \wedge \tau}$ is always well defined.

Stopped Martingales are also Martingales

If $\{M_n\}$ is a martingale with respect to $\{\mathcal{F}_n\}$, then the stopped process $\{M_{n \wedge \tau}\}$ is also a martingale with respect to $\{\mathcal{F}_n\}$.

Proof.

We can always write

$$M_{n \wedge \tau} = M_{n+1} \mathbf{1}_{\{\tau \geq n+1\}} + M_{n \wedge \tau} \mathbf{1}_{\{\tau \leq n\}}.$$

Since $\mathbf{1}_{\{\tau \geq n+1\}} = 1 - \mathbf{1}_{\{\tau \leq n\}}$, we see that $\mathbf{1}_{\{\tau \geq n+1\}}$ is $\mathcal{F}_n$ measurable. Thus

$$\begin{aligned} \mathsf{E}(M_{n+1}\mathbf{1}_{\{\tau \geq n+1\}} \mid \mathcal{F}_n) &= \mathsf{E}(M_{n+1} \mid \mathcal{F}_n)\mathbf{1}_{\{\tau \geq n+1\}} \\ &= M_n \mathbf{1}_{\{\tau \geq n+1\}} \\ &= M_{n\wedge\tau}\mathbf{1}_{\{\tau \geq n+1\}}. \end{aligned}$$

$$\begin{aligned} \mathsf{E}(M_{(n+1)\wedge\tau} | \mathcal{F}_n) &= \mathsf{E}(M_{n+1}\mathbf{1}_{\{\tau \geq n+1\}} \mid \mathcal{F}_n) + M_{n\wedge\tau}\mathbf{1}_{\{\tau \leq n\}} \\ &= M_{n\wedge\tau}\mathbf{1}_{\{\tau \geq n+1\}} + M_{n\wedge\tau}\mathbf{1}_{\{\tau \leq n\}} \\ &= M_{n\wedge\tau}, \end{aligned}$$

proving that $\{M_{n\wedge\tau}\}$ is a martingale with respect to $\{\mathcal{F}_n\}$. □

Ruin Probabilities Again

Let us reconsider the simple symmetric random walk:

$$S_n = \sum_{i=1}^{n} X_i$$

for $n \geq 1$, where each X_i is an independent fair coin toss, taking values -1 or 1 with probability $1/2$. Define the stopping time

$$\tau = \min\{n > 0 : S_n = A \text{ or } S_n = -B\},$$

which is measurable with respect to the filtration $\mathcal{F}_n$.

The Probability

Since τ is almost surely finite, the stopped process $S_{n\wedge\tau}$ converges to S_τ as $n \to \infty$:

$$\lim_{n\to\infty} S_{n\wedge\tau} = S_\tau.$$

Recall that S_n is a martingale, so by the stopped process property, $S_{n\wedge\tau}$ is also a martingale.

Moreover, since the process stops when it reaches either A or $-B$, we have

$$|S_{n\wedge\tau}| \le \max(A, B)$$

for all n. This uniform bound allows us to apply the dominated convergence theorem, yielding

$$\lim_{n\to\infty} \mathsf{E}(S_{n\wedge\tau}) = \mathsf{E}\left(\lim_{n\to\infty} S_{n\wedge\tau}\right) = \mathsf{E}(S_\tau).$$

We can then conclude that

$$\mathsf{E}(S_\tau) = \lim_{n\to\infty} \mathsf{E}(S_{n\wedge\tau}) = S_{0\wedge\tau} = 0.$$

Since we can also write

$$\mathsf{E}(S_\tau) = \mathsf{P}(S_\tau = A)A + (1 - \mathsf{P}(S_\tau = A))(-B) = 0,$$

we get that

$$\mathsf{P}(S_\tau = A) = \frac{B}{A+B}.$$

The Expectation

Furthermore, define $M_n = S_n^2 - n$ as before. Since for the simple random walk $\mathsf{E}(X_n) = 0$ and $\mathsf{E}(X_n^2) = 1$, this implies that $\{M_n\}$ is a martingale. Furthermore,

$$|M_{n\wedge\tau}| \le \max(A^2, B^2) + \tau.$$

Because $\mathsf{E}(\tau)\infty$, we can apply again the dominated convergence theorem to show

$$\lim_{n\to\infty} \mathsf{E}(M_{n\wedge\tau}) = \mathsf{E}\left(\lim_{n\to\infty} M_{n\wedge\tau}\right) = \mathsf{E}(M_\tau),$$

but also

$$\mathsf{E}(M_\tau) = \lim_{n\to\infty} \mathsf{E}(M_{n\wedge\tau}) = M_{0\wedge\tau} = 0.$$

Since we also have that

$$\mathsf{E}(M_\tau) = A^2\frac{B}{A+B} + B^2\frac{A}{A+B} - \mathsf{E}(\tau) = 0,$$

we conclude that

$$\mathsf{E}(\tau) = AB.$$