Prof. Lorenzo Naranjo Fall 2025

Discount Factors in Continuous Time

Price Processes

We work in a probability space $(\Omega, \mathcal{F}, \mathrm{P})$ where uncertainty is driven by standard Brownian motions defining a filtration $\{\mathcal{F}_t\}$. We model the price of risky assets as diffusions

$$\frac{\mathrm{d}S_t}{S_t} = \mu_S(t)\,\mathrm{d}t + \sigma_S(t)\,\mathrm{d}B_S(t),$$

where the drift $\mu_S(t)$ and the volatility of returns $\sigma_S(t)$ are $\mathcal{F}_t$-adapted processes. Most often for simplicity I will just write

$$\frac{\mathrm{d}S}{S} = \mu_S\,\mathrm{d}t + \sigma_S\,\mathrm{d}B_S.$$

Furthermore, in the following I use $\mathrm{E}_t(\cdot)$ to denote $\mathrm{E}(\cdot \mid \mathcal{F}_t)$, so that

$$\mathrm{E}_t\left(\frac{\mathrm{d}S_t}{S_t}\right) = \mu_S(t)\,\mathrm{d}t.$$

The idea of using the conditional expectation in this way is to extract the drift part of the diffusion process, so many times times I will just write

$$\mathrm{E}\left(\frac{\mathrm{d}S}{S}\right) = \mu_S\,\mathrm{d}t,$$

with the understanding that the expectation is there to extract the drift of the diffusion.

The total instantaneous return of an asset is given by

$$\frac{\mathrm{d}S + D\,\mathrm{d}t}{S} = \frac{\mathrm{d}S}{S} + \frac{D}{S}\,\mathrm{d}t,$$

where $\frac{D}{S}$ is the dividend yield. The dividend yield of a stock determines the number of new shares that the dividend process generates. Unlike cash dividends, the dividend yield acts as if dividends are reinvested in the stock. Thus, we can always work with dividend-reinvested assets instead. To see this, let

$$\frac{\mathrm{d}X}{X} = \frac{D}{S}\,\mathrm{d}t. \tag{1}$$

Here X represents the number of new shares accruing to the owner of the stock determined by the dividend yield. We can derive the previous expression by noting that $XD\,\mathrm{d}t$ represents the total amount of cash provided by the dividends during the interval $\mathrm{d}t$. This amount of cash would allow to purchase $\frac{XD\,\mathrm{d}t}{S}$ new shares, so that

$$\mathrm{d}X = \frac{XD\,\mathrm{d}t}{S} = X\frac{D}{S}\,\mathrm{d}t,$$

which yields equation (1). Note that

$$X_t = X_0 \exp\left(\int_0^t \frac{D_u}{S_u}\,\mathrm{d}u\right), \tag{2}$$

so the total number of shares grows exponentially with a growth rate equal to the dividend yield of the asset. In other words, X_t keeps track of the total number of shares at each point in time.

The dividend-reinvested asset price V is then

$$V = XS,$$

where V denotes the total value of this investment given by the number of shares times the price per share. We can find the dynamics of V by applying Ito's lemma:

$$\frac{\mathrm{d}V}{V} = \frac{\mathrm{d}S}{S} + \frac{\mathrm{d}X}{X} = \frac{\mathrm{d}S}{S} + \frac{D}{S}\,\mathrm{d}t.$$

Not surprisingly, the dynamics of P are characterized by capital gains and a dividend yield.

We will assume that there is a risk-free rate of return r. We do not always assume that r is constant, as it could depend on time t, ω or other state variables. We assume that there is a money-market account β that earns a risk-free rate. If we start with β_0 in the account, we must have that

$$\frac{d\beta}{\beta} = r\,\mathrm{d}t. \tag{3}$$

We can solve for β to find,

$$\beta_t = \beta_0 \exp\left(\int_0^t r_s ds\right). \tag{4}$$

As mentioned before, we do not assume that r is constant. In many applications, the risk-free rate follows a diffusion such that

$$\mathrm{d}r_t = \mu_r(t)\,\mathrm{d}t + \sigma_r(t)\,\mathrm{d}B_r(t).$$

From Discrete to Continuous Time

We want to derive a stochastic discount factor for discounting risky cash flows in continuous time. To bridge discrete and continuous time, we first build a dividend-reinvested asset.

Consider an asset with price S_t and dividend yield δ_t known at time t. If we reinvest dividends, our total shares grow multiplicatively. Starting with one share at time 0, at time t we hold

$$X_t = \prod_{i=0}^{t-1}(1+\delta_i)$$

shares. The total value of our dividend-reinvested investment is then

$$V_t = X_t S_t.$$

This discrete-time construction parallels the continuous-time formula in equation (2), where the product becomes an integral in the exponent.

Now, if Λ_t is a strictly positive discount factor it must be the case that

$$\Lambda_t V_t = \mathrm{E}_t(\Lambda_{t+1} V_{t+1}),$$

or

$$\mathrm{E}_t(\Lambda_{t+1} V_{t+1} - \Lambda_t V_t) = 0. \tag{5}$$

If we denote the time interval by Δt, equation (5) becomes

$$\mathrm{E}_t(\Lambda_{t+\Delta t} V_{t+\Delta t} - \Lambda_t V_t) = 0.$$

If we now let $\Delta t \to 0$, the previous expression implies

$$\mathrm{E}_t\, \mathrm{d}(\Lambda_t V_t) = 0,$$

which we will usually just write as

$$\mathrm{E}\, \mathrm{d}(\Lambda V) = 0. \tag{6}$$

The previous expression asserts that the discounted dividend-reinvested price process is a local martingale, so the (conditional) expectation is just saying that the drift should be zero. We will discuss later situations in which this local martingale is in fact a martingale.

Since $\Lambda > 0$, we have that

$$\begin{aligned}
\frac{\mathrm{d}(\Lambda V)}{\Lambda V} &= \frac{\mathrm{d}V}{V} + \frac{\mathrm{d}\Lambda}{\Lambda} + \left(\frac{\mathrm{d}\Lambda}{\Lambda}\right)\left(\frac{\mathrm{d}V}{V}\right) \\
&= \frac{\mathrm{d}S}{S} + \frac{\mathrm{d}\Lambda}{\Lambda} + \left(\frac{\mathrm{d}\Lambda}{\Lambda}\right)\left(\frac{\mathrm{d}S}{S}\right) + \frac{D}{S}\,\mathrm{d}t \\
&= \frac{\mathrm{d}(\Lambda S)}{\Lambda S} + \frac{D}{S}\,\mathrm{d}t.
\end{aligned}$$

Thus, $\mathrm{E}\, \mathrm{d}(\Lambda V) = 0$ is equivalent to

$$\mathrm{E}\, \mathrm{d}(\Lambda S) + \Lambda D\, \mathrm{d}t = 0. \tag{7}$$

Equation (7) is an alternative to equation (6) which makes explicit the dividend process in pricing the asset.

An SDF in Continuous Time

Let's start computing the discounted process for the money market account β defined earlier. Remember that

$$\frac{\mathrm{d}\beta}{\beta} = r\,\mathrm{d}t.$$

Thus,

$$\mathrm{d}(\Lambda\beta) = \Lambda d\beta + \beta\,\mathrm{d}\Lambda.$$

The pricing equation (7) implies that $\mathsf{E}\, d(\Lambda\beta) = 0$, so that

$$\mathsf{E}(\Lambda\, \mathrm{d}\beta + \beta\, \mathrm{d}\Lambda) = 0,$$

or

$$\mathsf{E}\left(\frac{\mathrm{d}\Lambda}{\Lambda}\right) = -\,\mathsf{E}\left(\frac{\mathrm{d}\beta}{\beta}\right) = -r\,\mathrm{d}t.$$

Thus, the drift of the SDF in continuous time determines the equilibrium continuously-compounded risk-free rate. Remember that r need not be deterministic but an adapted process to the filtration of the probability space.

Applying Ito's lemma now to ΛS we find that

$$\mathrm{d}(\Lambda S) = S\,\mathrm{d}\Lambda + \Lambda\,\mathrm{d}S + \mathrm{d}\Lambda\,\mathrm{d}S,$$

or

$$\frac{\mathrm{d}(\Lambda S)}{\Lambda S} = \frac{\mathrm{d}\Lambda}{\Lambda} + \frac{\mathrm{d}S}{S} + \left(\frac{\mathrm{d}\Lambda}{\Lambda}\right)\left(\frac{\mathrm{d}S}{S}\right).$$

Taking expectations both sides, equation (7) implies that

$$\mathsf{E}\left(\frac{\mathrm{d}S}{S}\right) + \frac{D}{S}\,\mathrm{d}t = r\,\mathrm{d}t - \left(\frac{d\Lambda}{\Lambda}\right)\left(\frac{\mathrm{d}S}{S}\right).$$

Fundamental Pricing Equation

Consider an asset S that follows a diffusion

$$\frac{\mathrm{d}S}{S} = \mu\,\mathrm{d}t + \sigma\,\mathrm{d}B.$$

If the asset pays a dividend yield $\delta = D/S$, and there are no arbitrage opportunities, it must be the case that

$$(\mu + \delta - r)\,\mathrm{d}t = -\left(\frac{\mathrm{d}\Lambda}{\Lambda}\right)\left(\frac{\mathrm{d}S}{S}\right). \tag{8}$$

In words, the risk-premium of the asset equals minus the covariance of the SDF and the asset's returns.