Derivatives Pricing in Continuous Time

Price Processes

Let's start considering a non-dividend paying stock S over the time interval $[0,T]$. Of course, a stock that does not pay dividends forever is a bubble but we will focus on a finite period of time in which the stock does not pay dividends. The stock price process follows a geometric Brownian of the form

$$\frac{\mathrm{d}S}{S} = \mu\,\mathrm{d}t + \sigma\,\mathrm{d}B,$$

and there is a money-market account β that grows at the risk-free rate r such that

$$\frac{\mathrm{d}\beta}{\beta} = r\,\mathrm{d}t.$$

Since our objective is to price derivatives written on S with payoffs given by some function of the stock price at time T, we can write the discount factor as

$$\frac{\mathrm{d}\Lambda}{\Lambda} = -r\,\mathrm{d}t - \lambda\,\mathrm{d}B.$$

The stochastic part of the discount factor that matters for this application is the one that is perfectly correlated with the stock price process. For the moment, we assume that μ, r, σ and λ are all adapted process to the filtration on which all Brownian motions are adapted.

The pricing equation implies that λ should be the instantaneous Sharpe ratio of the stock price so that

$$\lambda = \frac{\mu - r}{\sigma}.$$

The Risk-Neutral Measure

Girsanov's theorem allows us to create Brownian motions under a different measure by using strictly positive martingales. The risk-neutral measure is a particular measure created by using the process $\mathcal{E} = \Lambda\beta$. Ito's lemma implies that

$$\frac{\mathrm{d}\mathcal{E}}{\mathcal{E}} = \frac{\mathrm{d}\Lambda}{\Lambda} + \frac{\mathrm{d}\beta}{\beta} = -\lambda\,\mathrm{d}B.$$

We can then compute

$$\mathcal{E}_t = \mathcal{E}_0 e^{-\int_0^t \lambda_s dB_s},$$

which is strictly positive as long $\mathcal{E}_0 > 0$. Note that many authors normalize $\Lambda_0 = 1$ and $\beta_0 = 1$ so that $\mathcal{E}_0 = 1$.

The previous expression implies that $\{\mathcal{E}_t\}$ is a strictly positive local martingale. If $\{\mathcal{E}_t\}$ is actually a martingale, we can create a new measure P^* such that

$$\frac{\mathrm{d\,P}^*}{d\,\mathrm{P}} = \frac{\mathcal{E}_T}{\mathcal{E}_0}.$$

Thus, for any $\mathcal{F}_t$-adapted process $\{X_t\}$ we have that

$$\mathrm{E}^*(X_T) = \mathrm{E}\left(\frac{\mathrm{d\,P}^*}{\mathrm{d\,P}} X_T\right) = \mathrm{E}\left(\frac{\mathcal{E}_T}{\mathcal{E}_0} X_T\right).$$

There are many models in which $\{\mathcal{E}_t\}$ is a proper martingale. For example, if λ is constant, then it is not hard to show that $\{\mathcal{E}_t\}$ is a martingale. In the following, we assume that $\{\lambda_t\}$ is such that $\{\mathcal{E}_t\}$ is a strictly positive martingale.

Girsanov's theorem then implies that for $0 \leq t \leq T$ we have that

$$B_t^* = B_t - \int_0^t \left(\frac{\mathrm{d}\mathcal{E}_s}{\mathcal{E}_s}\right)(\mathrm{d}B_s)$$

is a P^*-Brownian motion. If $\mathcal{E} = \Lambda\beta$, we can compute

$$\left(\frac{\mathrm{d}\mathcal{E}}{\mathcal{E}}\right)(\mathrm{d}B) = \left(\frac{\mathrm{d}\Lambda}{\Lambda} + \frac{\mathrm{d}\beta}{\beta}\right)(\mathrm{d}B) = -\lambda\,\mathrm{d}t.$$

Thus, we can write the new Brownian motion in differential form:

$$\mathrm{d}B^* = \mathrm{d}B + \lambda\,\mathrm{d}t.$$

The dynamics of S under P^* are then given by

$$\frac{\mathrm{d}S}{S} = \mu\,\mathrm{d}t + \sigma(\mathrm{d}B^* - \lambda\,\mathrm{d}t) = (\mu - \lambda\sigma)\,\mathrm{d}t + \sigma\,\mathrm{d}B^*. \tag{1}$$

Thus, we have that

$$\frac{\mathrm{d}S}{S} = r\,\mathrm{d}t + \sigma\,\mathrm{d}B^*.$$

The previous expression implies that the drift of the stock is just the risk-free rate under P^*. Consider now another asset V exposed to the same Brownian motion B,

$$\frac{\mathrm{d}V}{V} = \mu_V\,\mathrm{d}t + \sigma_V\,\mathrm{d}B. \tag{2}$$

It must also be the case that

$$\lambda = \frac{\mu - r}{\sigma} = \frac{\mu_V - r}{\sigma_V},$$

so that

$$\frac{\mathrm{d}V}{V} = r\,\mathrm{d}t + \sigma\,\mathrm{d}B^*. \tag{3}$$

Thus, all assets under P^* earn the same rate of return equal to the risk-free rate. This is why we call the measure P^* the **risk-neutral measure**. In a risk-neutral world, all investors are happy discounting all cash flows at the risk-free rate.

If ΛV is a martingale it must be the case that

$$\lambda_0 V_0 = \mathsf{E}(\Lambda_T V_T).$$

Thus,

$$V_0 = \mathsf{E}\left(\frac{\Lambda_T\beta_T}{\Lambda_0\beta_0}\frac{\beta_0}{\beta_T}V_T\right) = \mathsf{E}\left(\frac{\mathcal{E}_T}{\mathcal{E}_0}e^{-\int_0^T r_s ds}V_T\right) = \mathsf{E}^*\left(e^{-\int_0^T r_s ds}V_T\right). \tag{4}$$

Therefore, we can value any asset by discounting expected cash flows at the risk-free rate of return.

Example 1. The price of a zero-coupon bond paying 1 unit of consumption at time T is just

$$P(T) = \mathsf{E}^*\left(e^{-\int_0^T r_s ds}\right),$$

since $V_T = 1$. Therefore, $e^{-\int_0^T r_s ds}$ acts like a discount factor under the risk-neutral measure.

□

Example 2. A futures contract is an obligation to purchase or sell an asset S for a pre-specified price namely the futures price at a specific date T in the future. The key feature of futures contracts is that the gains or losses are realized daily. Also, to buy or sell a futures there is no cash outflow. Even though in real markets investors need to deposit a small margin, for the purpose of pricing the futures we can assume that the margin amount is negligible.

Therefore, if we denote by dF the futures gains or losses in a long position from t to $t + dt$, it must be the case that

$$\mathsf{E}(\Lambda\, \mathrm{d}F) = 0,$$

since no cash is required to obtain a potential gain or loss of dF during the period. We can then re-write the previous expression for $0 \leq t \leq T$ as

$$0 = \mathsf{E}\left(\frac{\Lambda_t \beta_t}{\Lambda_0 \beta_0}\, \mathrm{d}F\right) = \mathsf{E}^*(\mathrm{d}F).$$

Thus, under the risk-neutral measure the futures price process must be a local martingale. For many models, we can actually write that the futures price is a martingale under the risk-neutral measure, implying

$$F(T) = \mathsf{E}^*(S_T).$$

Even though sometimes it might be hard to show that the futures is a P^*-martingale, we can always compute $\mathsf{E}^*(S_T)$ and verify that the futures satisfy the local martingale property. □

Example 3. The forward price $\varphi(T)$ is the delivery price in a forward contract expiring at time T such that the value of the contract is zero, i.e.,

$$\mathsf{E}^*\left(e^{-\int_0^T r_s ds}(S_T - \varphi(T))\right) = 0.$$

Therefore,

$$\varphi(T) = \frac{\mathsf{E}^*\left(e^{-\int_0^T r_s ds} S_T\right)}{\mathsf{E}^*\left(e^{-\int_0^T r_s ds}\right)} = \frac{\mathsf{Cov}^*\left(e^{-\int_0^T r_s ds}, S_T\right)}{B(T)} + F(T).$$

The forward price is equal to the futures price plus the risk-neutral covariance between the risk-neutral discount factor and the underlying asset. Thus, the forward price is equal to the futures price only when this covariance is zero. □

The Black-Scholes Model

The Black-Scholes formula to price options is one of the most important accomplishments in finance. A European call option gives its buyer the right but not the obligation to purchase an asset for a pre-determined price K at a future date T. Thus, the buyer of the call option pays K to receive a stock worth S_T only when $S_T > K$.

In their original model, Black and Scholes (1973) assumes that all parameters are constant. This implies that

$$\beta_T = \beta_0 e^{rT},$$

and

$$V_0 = e^{-rT}\,\mathsf{E}^*(V_T).$$

In the Black-Scholes model, the stock price process under the risk-neutral measure is

$$\frac{\mathrm{d}S}{S} = r\,\mathrm{d}t + \sigma\,\mathrm{d}B^*.$$

We can solve for S_T to find

$$S_T = S_0 e^{(r-\frac{1}{2}\sigma^2)T + \sigma B_T^*}.$$

Thus, under the risk-neutral measure $\ln(S_T)$ is normally distributed with mean

$$\mathsf{E}\ln(S_T) = \ln(S_0) + \left(r - \tfrac{1}{2}\sigma^2\right)T,$$

and variance

$$\mathsf{V}\ln(S_T) = \sigma^2 T.$$

Example 4. The risk-neutral probability that the stock price S_T is greater than K at time T is

$$\begin{aligned}
\mathsf{P}^*(S_T > K) &= \mathsf{P}^*(\ln(S_T) > \ln(K)) \\
&= \mathsf{P}^*\left(Z > \frac{\ln(K) - \ln(S_0) - \left(r - \frac{1}{2}\sigma^2\right)T}{\sigma^2 T}\right) \\
&= \mathsf{P}^*\left(Z < \frac{\ln(S/K) + \left(r - \frac{1}{2}\sigma^2\right)T}{\sigma^2 T}\right),
\end{aligned}$$

where Z denotes a standard normally distributed random variable. In the Black-Scholes model, we typically write

$$d_2 = \frac{\ln(S/K) + \left(r - \frac{1}{2}\sigma^2\right)T}{\sigma^2 T},$$

and $N(d) = \mathsf{P}^*(Z < d)$, so that $\mathsf{P}^*(S_T > K) = N(d_2)$. □

For a given event $A \in \mathcal{F}$, the indicator function $\mathbf{1}_{\{A\}}(\omega)$ is equal to 1 if $\omega \in A$ and 0 otherwise. Thus, $\mathbf{1}_{\{S_T > K\}}$ is equal to 1 whenever $S_T > K$ and zero otherwise. The payoff of a call option can then be defined as

$$\text{Call Payoff} = (S_T - K)\mathbf{1}_{\{S_T > K\}} = S_T \mathbf{1}_{\{S_T > K\}} - K\mathbf{1}_{\{S_T > K\}}.$$

The price of a call must then be given by

$$C_0 = \mathsf{E}\left(\frac{\Lambda_T}{\Lambda_0}(S_T - K)\mathbf{1}_{\{S_T > K\}}\right) = \mathsf{E}\left(\frac{\Lambda_T}{\Lambda_0} S_T \mathbf{1}_{\{S_T > K\}}\right) - K\,\mathsf{E}\left(\frac{\Lambda_T}{\Lambda_0}\mathbf{1}_{\{S_T > K\}}\right). \tag{5}$$

To compute the first expectation in (5), a nice trick is to realize that $\mathcal{E}^S = \Lambda S$ is a strictly positive martingale defining a new measure P^S such that

$$\frac{\mathrm{d}\,\mathsf{P}^S}{\mathrm{d}\,\mathsf{P}} = \frac{\mathcal{E}_T^S}{\mathcal{E}_0^S}.$$

Thus,

$$\mathsf{E}\left(\frac{\Lambda_T}{\Lambda_0} S_T \mathbf{1}_{\{S_T > K\}}\right) = S_0\,\mathsf{E}\left(\frac{\Lambda_T S_T}{\Lambda_0 S_0}\mathbf{1}_{\{S_T > K\}}\right) = S_0\,\mathsf{E}^S\left(\mathbf{1}_{\{S_T > K\}}\right) = S_0\,\mathsf{P}^S(S_T > K).$$

To compute the second expectation in (5) we can just use the risk-neutral measure

$$\mathsf{E}\left(\frac{\Lambda_T}{\Lambda_0}\mathbf{1}_{\{S_T > K\}}\right) = \frac{\beta_0}{\beta_T}\,\mathsf{E}\left(\frac{\Lambda_T \beta_T}{\Lambda_0 \beta_0}\mathbf{1}_{\{S_T > K\}}\right) = e^{-rT}\,\mathsf{E}^*\left(\mathbf{1}_{\{S_T > K\}}\right) = e^{-rT}\,\mathsf{P}^*(S_T > K).$$

The price of the call can then be written as

$$C_0 = S_0\,\mathsf{P}^S(S_T > K) - Ke^{-rT}\,\mathsf{P}^*(S_T > K).$$

To compute $\mathsf{P}^S(S_T > K)$, we know that

$$B_t^S = B_t - \int_0^t \left(\frac{\mathrm{d}\mathcal{E}^S}{\mathcal{E}^S}\right)(\mathrm{d}B) = B_t + (\lambda - \sigma)t$$

is a Brownian motion under P^S. Thus,

$$\frac{\mathrm{d}S}{S} = (r + \sigma^2)\,\mathrm{d}t + \sigma\,\mathrm{d}B^S.$$

We can follow the steps in Example 4 to conclude that

$$\mathsf{P}^S(S_T > K) = N(d_1),$$

where

$$d_1 = \frac{\ln(S/K) + \left(r + \frac{1}{2}\sigma^2\right)T}{\sigma^2 T}.$$

To price a European put option we can proceed in a similar way. Remember that a European put option gives it's buyer the right but not the obligation to sell an asset for a pre-determined price K at a future date T. Therefore, the payoff of the European put at maturity is

$$\text{Put Payoff} = K\mathbf{1}_{\{S_T<k\}} - S_T\mathbf{1}_{\{S_T<k\}}.$$

The price P_0 of the put today is then given by

$$P_0 = Ke^{-rT}\,\mathsf{P}^*(S_T < K) - S\,\mathsf{P}^S(S_T < K).$$

Thus, $\mathsf{P}^*(S_T < K) = 1 - N(d_2) = N(-d_2)$ and $\mathsf{P}^S(S_T < K) = 1 - N(d_1) = N(-d_1)$. We can summarize these results in the following property.

The Black-Scholes Model

In the Black-Scholes model, the prices C and P of European call and put options, respectively, are given by

$$C = SN(d_1) - Ke^{-rT}N(d_2),$$
$$P = Ke^{-rT}N(-d_2) - SN(-d_1),$$

where

$$d_1 = \frac{\ln(S/K) + \left(r + \frac{1}{2}\sigma^2\right)T}{\sigma^2 T},$$

$$d_2 = \frac{\ln(S/K) + \left(r - \frac{1}{2}\sigma^2\right)T}{\sigma^2 T},$$

and $N(d)$ denotes the cumulative probability than a standard normal random variable is less than d.

Partial Differential Equations in the Black-Scholes Model

The Black-Scholes formula was originally derived as the solution of a partial differential equation (PDE). It is indeed the case that any asset in the Black-Scholes model must satisfy the same PDE. Consider a derivative V that pays $f(S_T)$ at time T.

If V is a function of S and t, then Ito's lemma implies that

$$\begin{aligned} dV &= \frac{\partial V}{\partial S} dS + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}(dS)^2 + \frac{\partial V}{\partial t} dt \\ &= \left(rS\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t} \right) dt + \sigma S \frac{\partial V}{\partial S} dB^*. \end{aligned}$$

Equation (3) then implies that any derivative written on S must satisfy the following partial differential equation

$$rS\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t} = rV.$$

It is in theory possible to solve the partial differential equation subject to a terminal value to price any derivative like a European call or put option written on the stock. In practice, it is easier to use a change of measure to find the value of the derivative.

References

Black, Fischer, and Myron Scholes. 1973. "The Pricing of Options and Corporate Liabilities." *Journal of Political Economy* 81 (3): 637–54.