

Problem Set 1 Solutions

Problem 1. Exactly one year ago, Maria paid \$1,020 for a corporate bond with a face value of \$1,000. During the year she collected two semiannual coupon payments totaling \$60 in interest. Today, with market yields slightly higher than when she bought it, she sold the bond for \$995. What holding-period return did Maria earn over the year?

Solution. The holding-period return adds the coupons to the price change and divides by the purchase price:

$$r = \frac{P_1 - P_0 + C}{P_0} = \frac{995 - 1,020 + 60}{1,020} = \frac{35}{1,020} = 3.43\%.$$

The \$60 in coupons more than offsets the \$25 capital loss.

Problem 2. Nadia is evaluating a one-year stake in a private real-estate deal. A year from now the deal will be wound up and its proceeds paid out: if the anchor tenant renews its lease, investors receive \$260,000; if the tenant walks away, they receive \$120,000. Nadia judges the two outcomes equally likely. Her only alternative is to hold one-year Treasury bills, which pay 4% with certainty. She values the stake by discounting its expected proceeds at the return she requires.

- Nadia demands a risk premium of 7% over the T-bill rate to bear this uncertainty, so her required return is $4\% + 7\% = 11\%$. What is the most she should be willing to pay for the stake?
- Suppose the stake is offered at exactly the price you found in part a. What expected rate of return would Nadia earn?
- Now suppose Nadia instead demands a risk premium of 10%, so her required return is $4\% + 10\% = 14\%$. What is the most she should be willing to pay?

Solution. The expected proceeds are

$$E[X] = 0.5 \times 260,000 + 0.5 \times 120,000 = 190,000.$$

a. Discounting at 11%, the most Nadia should pay is

$$P = \frac{190,000}{1.11} = \$171,171.17.$$

b. At that price her expected return is

$$\frac{190,000}{171,171.17} - 1 = 11\%,$$

exactly her required return: 4% for time plus 7% for bearing the risk. Pricing at the required return makes the expected return equal to it.

c. Discounting at 14%,

$$P = \frac{190,000}{1.14} = \$166,666.67.$$

A higher risk premium lowers the price she is willing to pay for the same expected proceeds, by about \$4,500.

Problem 3. Shares of Delta Robotics trade today at \$60, and the company just paid its annual dividend of \$1.20. An analyst covering the stock lays out three scenarios for the coming year, each with its own forecast for the year-end share price and the dividend paid over the year:

State of the Market	Probability	Dividend	Price
Expansion	0.25	\$2.00	\$75
Steady growth	0.45	\$1.50	\$62
Downturn	0.30	\$0.50	\$48

Compute the mean and standard deviation of the one-year return on Delta Robotics stock.

Solution. The return in each state is $r = (D_1 + P_1 - P_0)/P_0$ with $P_0 = 60$. The \$1.20 dividend already paid does not enter the return over the coming year.

State of the Market	Probability	Return
Expansion	0.25	$(2.00 + 75 - 60)/60 = 28.33\%$
Steady growth	0.45	$(1.50 + 62 - 60)/60 = 5.83\%$
Downturn	0.30	$(0.50 + 48 - 60)/60 = -19.17\%$

The mean return is

$$E[r] = 0.25 \times 28.33\% + 0.45 \times 5.83\% + 0.30 \times (-19.17\%) = 3.96\%.$$

The variance is

$$V[r] = 0.25(0.2833 - 0.0396)^2 + 0.45(0.0583 - 0.0396)^2 + 0.30(-0.1917 - 0.0396)^2 \\ = 0.03105,$$

so the standard deviation is $\sqrt{0.03105} = 17.62\%$.

Problem 4. Two companies serve the outdoor-recreation market: RainCo, which sells rain gear, and SunCo, which sells sunscreen and beach equipment. Next year's weather can turn out wet, normal, or dry, and you assess the probabilities and the resulting year-end share prices as follows. RainCo trades at \$20 today and SunCo at \$25; neither pays a dividend.

Weather	Probability	RainCo price	SunCo price
Wet	0.3	\$30	\$26
Normal	0.4	\$22	\$25
Dry	0.3	\$16	\$28

- Compute the expected return of each stock.
- Compute the standard deviation of returns for each stock.
- Compute the covariance and the correlation between the two stocks' returns.

Solution. Neither stock pays a dividend, so the return in each state is $P_1/P_0 - 1$:

Weather	Probability	RainCo return	SunCo return
Wet	0.3	50%	4%
Normal	0.4	10%	0%
Dry	0.3	−20%	12%

a. Expected returns:

$$E[r_R] = 0.3 \times 50\% + 0.4 \times 10\% + 0.3 \times (-20\%) = 13\%,$$

$$E[r_S] = 0.3 \times 4\% + 0.4 \times 0\% + 0.3 \times 12\% = 4.8\%.$$

b. Variances and standard deviations:

$$V[r_R] = 0.3(0.50 - 0.13)^2 + 0.4(0.10 - 0.13)^2 + 0.3(-0.20 - 0.13)^2 = 0.0741,$$

$$V[r_S] = 0.3(0.04 - 0.048)^2 + 0.4(0 - 0.048)^2 + 0.3(0.12 - 0.048)^2 = 0.002496,$$

$$\text{so } \sigma_R = \sqrt{0.0741} = 27.22\% \text{ and } \sigma_S = \sqrt{0.002496} = 5.00\%.$$

c. The covariance is

$$\begin{aligned} \text{Cov}[r_R, r_S] &= 0.3(0.37)(-0.008) + 0.4(-0.03)(-0.048) + 0.3(-0.33)(0.072) \\ &= -0.000888 + 0.000576 - 0.007128 = -0.00744, \end{aligned}$$

and the correlation is

$$\rho = \frac{-0.00744}{0.2722 \times 0.0500} = -0.547.$$

RainCo does well when it is wet and SunCo when it is dry, so the two returns tend to move in opposite directions.

Problem 5. Your preferences over end-of-year wealth are described by the logarithmic utility function $u(W) = \ln W$, and your savings currently amount to \$13,000. All of that wealth is tied up in a single volatile position that, over the coming year, is equally likely to gain \$3,000 or lose \$3,000. For a fee of \$400 you can buy a hedge today that locks in your wealth and removes the risk entirely.

a. Do you buy the hedge, or do you carry the position unhedged? Support your answer by comparing expected utilities.

- b. Suppose you carried the position unhedged, the bad outcome occurred, and your savings fell to \$10,000. You are now offered the very same position for the year ahead and the very same \$400 hedge. Do you buy the hedge this time? Explain what changed, given that neither the size of the gamble nor the price of the hedge did.

Solution.

- a. Unhedged, your wealth is \$16,000 or \$10,000 with equal probability:

$$E[u(W)] = 0.5 \ln 16,000 + 0.5 \ln 10,000 = 9.4453.$$

Hedged, your wealth is \$13,000 minus the fee, \$12,600 for sure, and $\ln 12,600 = 9.4415$. Expected utility is higher unhedged, so you do not buy the hedge. Equivalently, the certainty equivalent of the gamble is $e^{9.4453} = \$12,649.11$, so the most you would pay to remove the risk is $13,000 - 12,649.11 = \$350.89$, less than the \$400 fee.

- b. With \$10,000, unhedged wealth is \$13,000 or \$7,000:

$$E[u(W)] = 0.5 \ln 13,000 + 0.5 \ln 7,000 = 9.1632,$$

while hedged wealth is \$9,600 and $\ln 9,600 = 9.1695$. Now you buy the hedge. The certainty equivalent is $e^{9.1632} = \$9,539.39$, so you would pay up to $10,000 - 9,539.39 = \$460.61$ to remove the risk, more than the \$400 fee. What changed is your wealth. Log utility has decreasing absolute risk aversion: the same \$3,000 gamble is a larger share of a smaller fortune, so it hurts more to bear and you are willing to pay more to avoid it.

Problem 6. A vineyard owner's entire net worth of \$200,000 is invested in next season's crop. The weather over the growing season determines the outcome: with probability 5% a hailstorm destroys most of the harvest and her net worth falls to \$20,000; with probability 15% a late frost causes partial damage and her net worth falls to \$120,000; and with probability 80% the season is normal and her net worth stays at \$200,000. Her utility over net worth is logarithmic, $u(W) = \ln W$. A crop-insurance policy would pay \$180,000 after a hailstorm, \$80,000 after a frost, and nothing in a normal season. What is the most she would be willing to pay for this policy?

Solution. With the policy, her net worth is \$200,000 in every state (for example, 20,000 + 180,000 after a hailstorm), minus the premium π . The policy removes all risk, so the most she would pay makes her indifferent between 200,000 – π for sure and the uninsured gamble. Her expected utility without insurance is

$$E[u(W)] = 0.05 \ln 20,000 + 0.15 \ln 120,000 + 0.80 \ln 200,000 = 12.0143,$$

with certainty equivalent $e^{12.0143} = \$165,102.13$. The maximum premium is

$$\pi = 200,000 - 165,102.13 = \$34,897.87.$$

The expected payout of the policy is $0.05 \times 180,000 + 0.15 \times 80,000 = \$21,000$, so she is willing to pay about \$13,900 above the actuarially fair price.

Problem 7. A songwriter has died, and the rights to her music catalog, valued at \$500,000, are disputed by two former collaborators. Ana currently has no wealth, while Beto has \$20,000. Neither can prove sole authorship, so they can take the matter to court, but whoever is awarded the catalog will pay legal costs equal to 20% of its value. The judge must award the entire catalog to one of them; a split ruling is not possible. Both Ana and Beto have the same utility function over wealth,

$$U(W) = W^{1/2}.$$

- Ana and Beto both agree that Ana has a 70% chance of prevailing in court and Beto a 30% chance. Should they litigate, or is there a settlement that makes both better off? If a settlement exists, give one.
- Now suppose Ana and Beto are each overconfident, and both believe they have an 80% chance of winning. Does a mutually beneficial settlement still exist? Explain what drives the change.

Solution. Whoever wins the catalog keeps $500,000 \times (1 - 0.2) = \$400,000$ after legal costs. A settlement splits the full \$500,000 and avoids those costs.

- Ana's expected utility from litigating is

$$E[U_A] = 0.7\sqrt{400,000} + 0.3\sqrt{0} = 442.72,$$

with certainty equivalent $442.72^2 = \$196,000$. Beto's is

$$E[U_B] = 0.3\sqrt{420,000} + 0.7\sqrt{20,000} = 293.42,$$

with certainty equivalent $293.42^2 = \$86,094$, a gain of \$66,094 over his current wealth. A settlement that gives Ana more than \$196,000 and Beto more than \$66,094 makes both better off, and since $196,000 + 66,094 = 262,094 < 500,000$, many such settlements exist. Splitting the catalog 70/30 works: Ana receives \$350,000 and Beto \$150,000, and both are better off than in court. They should settle.

b. If Ana believes she wins with probability 0.8,

$$E[U_A] = 0.8\sqrt{400,000} = 505.96,$$

with certainty equivalent \$256,000. If Beto believes the same about himself,

$$E[U_B] = 0.8\sqrt{420,000} + 0.2\sqrt{20,000} = 546.74,$$

with certainty equivalent \$298,928, a gain of \$278,928. Together they require $256,000 + 278,928 = \$534,928$, more than the \$500,000 there is to split, so no settlement satisfies both and they litigate. Risk aversion and the 20% legal costs both favor settling, but each expects too much from court: their beliefs about winning add up to 160%, so the shares they think they are entitled to exceed what exists.