Statistics of Asset Returns

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These notes introduce the basic building blocks used to measure and describe the performance of an investment. We start by defining what it means to hold an asset and the cash flows it can generate, and use this to define the holding-period return (HPR), the basic unit of profitability in finance. We then treat returns as random variables and introduce their two most important moments, the expected return and the volatility, first using scenario analysis and then using historical stock and ETF data. The notes close by extending these ideas to portfolios of several assets, and by introducing the law of one price, the principle underlying portfolio construction, arbitrage, and the pricing of Exchange-Traded Funds.

Financial and Real Assets

Investing involves purchasing an asset today, which you may sell the next period, sell at some later date, or hold indefinitely. Even if you hold the asset indefinitely, we can always take the perspective of an investor who sells it and immediately repurchases it every period, which lets us evaluate its performance one period at a time regardless of your actual holding horizon.

The asset you purchase can be a **real asset**, like a house, or a **financial asset** like a stock or a bond. Financial assets are claims to parts of real assets, or to the income they generate. Because every financial asset is also someone else's liability, financial assets net to zero across the economy as a whole. Real assets are the only source of society's net wealth.

When you purchase an asset today and hold it for one period, you might also be entitled to a cash flow along the way. Many stocks in the U.S. pay dividends quarterly, while bonds typically pay coupons semi-annually. Real assets can generate cash flows too. For example, if you purchase a house and rent it, you will collect a monthly rent from your tenant.

The Rate of Return

In order to analyze the performance of your investment, it is useful to think in terms of profitability per dollar invested. Say you purchase an asset at time 0 for P_0 and sell it a period later for P_1. If the asset pays a cash flow, we assume it is paid just before you purchase or sell the asset. The **holding-period return (HPR)** from time 0 to time 1 is defined as

$$\text{HPR} = \frac{P_1 - P_0 + D_1}{P_0}.$$

The HPR can be decomposed as

$$\text{HPR} = \frac{P_1 - P_0}{P_0} + \frac{D_1}{P_0}.$$

The first term in the previous expression denotes the **capital gain** and the second term represents the **income yield**. Thus, the total return of investing in the asset depends on how much the price appreciates or depreciates, and the amount of cash flow paid during the holding period.

Example 1. Suppose you purchase today an index fund for \$100 per share. The fund will pay dividends over a year of \$4. If the price per share next year is \$110, your HPR amounts to

$$\text{HPR} = \frac{110 - 100 + 4}{100} = 14\%.$$

The total return of 14% consists of 10% in capital gains and 4% of dividend yield. □

To ease notation, in the following we denote by r_i the HPR of investing in asset i. Since there is considerable uncertainty about the future price and cash flows of the asset, its HPR is unknown today. A useful way to think about the uncertainty of r_i is to model it as a random variable in certain probability space (Ω, P).

The expectation of r_i is called the **expected return** of asset i, whereas the standard deviation of r_i is typically called the **volatility** or standard deviation of returns. I usually write

$$\mu_i = \mathrm{E}(r_i),$$
$$\sigma_i = \sqrt{\mathrm{V}(r_i)}.$$

Example 2. Suppose your expectations regarding a stock price are as follows:

State of the Market	Probability	HPR
Boom	0.35	44.5%
Normal	0.30	14%
Recession	0.35	-16.5%

We can compute the mean and standard deviation of the stock's HPR as

$$\begin{aligned}\mu &= 0.35 \times 0.445 + 0.3 \times 0.14 + 0.35 \times (-0.165) = 14\%,\\ \sigma &= \sqrt{0.35 \times (0.445 - 0.14)^2 + 0.3 \times (0.14 - 0.14)^2 + 0.35 \times (-0.165 - 0.14)^2}\\ &= 25.52\%.\end{aligned}$$

□

Stock Returns

The previous example assumed we knew the probabilities of each state of the market. In practice, we rarely know these probabilities, so we instead estimate the mean and volatility of returns from historical data.

Figure 1 plots the monthly returns of Apple (AAPL) since the stock was listed in December 1980. The figure shows that returns are volatile with periods in which volatility spikes and other periods of lower volatility. We can compute the average monthly return as

$$\bar{r} = \frac{1}{N}\sum_{t=1}^{N} r_t,$$

and sample standard deviation as

$$\hat{\sigma} = \sqrt{\frac{1}{N-1}\sum_{t=1}^{N}(r_t - \bar{r})^2}.$$

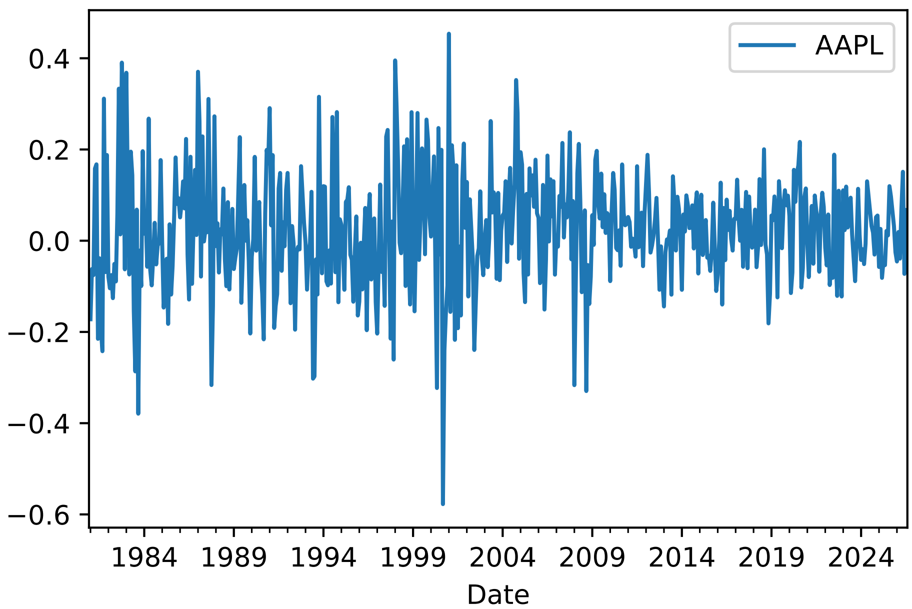

Figure 1: The figure shows the monthly returns of Apple (AAPL) since the stock was listed in December 1980.

If we assume that the monthly returns are independent and identically distributed, we can annualize the average monthly return and volatility as:

$$\bar{r}_{\text{Annual}} = 12 \times \bar{r}_{\text{Monthly}},$$
$$\hat{\sigma}_{\text{Annual}} = \sqrt{12} \times \hat{\sigma}_{\text{Monthly}}.$$

Using these expressions, we get for Apple that

	Monthly Estimate (%)	Annualized Estimate (%)
Mean	2.24	26.91
St. Dev.	12.49	43.27

Because the volatility is so high compared to the average return, estimates of average returns are typically very noisy and not very useful in practice to proxy for expected returns. Also, an average of monthly returns going back to 1980 is probably not a very good proxy of your *expectation* for returns going forward.

Portfolios

In finance, a **portfolio** consists of allocating a certain amount of wealth into different assets such as stocks, bonds, and real estate. Say you have two assets A and B. Denote by r_A and r_B the HPR of each asset, respectively. If you have a certain amount of wealth W, you can split it and invest W_A in asset A and the rest, $W_B = W - W_A$, in B. Each dollar invested in A yields $1 + r_A$ dollars next period whereas each dollar invested in B generates $1 + r_B$.

Denote by r the HPR of your portfolio. Using this notation, the total return of your investment next period is $(1 + r)W$. But this amount can be computed by considering the returns of the individual investments in A and B. Therefore, we must have that

$$(1 + r)W = (1 + r_A)W_A + (1 + r_B)W_B.$$

Since $W = W_A + W_B$, the previous expression can be simplified as

$$r = \frac{W_A}{W}r_A + \frac{W_B}{W}r_B. \tag{1}$$

In equation (1), the fractions W_A/W and W_B/W denote the proportion of wealth allocated to each asset. In finance, we call these fractions the **portfolio weights**. If we denote by w_A and w_B these portfolio weights, we can write expression (1) as

$$r = w_A r_A + w_B r_B.$$

An important thing to note is that the sum of the weights is by construction always equal to one.

Example 3. You have the following scenario analysis for the HPR of stocks X and Y:

Market	Probability	Stock X	Stock Y
Bull	0.3	40%	10%
Normal	0.5	15%	20%
Bear	0.2	-18%	-5%

Assume that of your \$10,000 portfolio, you invest \$8,000 in Stock X and \$2,000 in Stock Y.

To compute the expected return and standard deviation of the portfolio, we can first compute the HPR in each state of the world:

Market	Probability	Invested in X	Invested in Y	Total	HPR
Bull	0.3	11,200	2,200	13,400	34.0%
Normal	0.5	9,200	2,400	11,600	16.0%
Bear	0.2	6,560	1,900	8,460	-15.4%

Thus,

$$\begin{aligned}\mu_P &= 0.3 \times 0.34 + 0.5 \times 0.16 + 0.2 \times (-0.154) = 15.12\%,\\ \sigma_P &= \sqrt{0.3 \times (0.34 - 0.1512)^2 + 0.5 \times (0.16 - 0.1512)^2 + 0.2 \times (-0.154 - 0.1512)^2}\\ &= 17.14\%.\end{aligned}$$

□

The Law of One Price

The portfolio mathematics we just did relies on a very important assumption called the **law of one price** (LOOP). In competitive markets, LOOP guarantees that the price of a basket of stocks is equal to the sum of the prices of its constituents. This logic is at the heart of how **Exchange-Traded Funds** (ETF) operate, as the next example shows.

Example 4. An ETF is a type of investment fund that is traded on stock exchanges, similar to individual stocks. ETFs hold a diversified portfolio of assets, such as stocks, bonds, or commodities, which provides investors with broad exposure to specific markets or investment strategies.

ETF arbitrage is the mechanism that helps keep the market price of an ETF in line with its **Net Asset Value (NAV)**. **Authorized Participants (APs)**, typically large financial institutions, have the ability to create or redeem ETF shares in large blocks called **creation units**.

When the ETF market price is higher than the NAV, APs can buy the underlying securities of the ETF in the open market and then deliver them to the ETF issuer in exchange for new ETF shares. The AP can then sell these ETF shares at the higher market price, making a profit. This buying of underlying securities pushes their prices up, while the selling of new ETF shares pushes the ETF price down, bringing the two prices closer together.

When the ETF market price is lower than the NAV, APs can buy ETF shares in the open market and deliver them to the ETF issuer in exchange for the underlying securities. The AP can then sell these underlying securities at the higher NAV price, making a profit. This buying of ETF shares pushes their price up, while the selling of the underlying securities pushes their prices down, again bringing the two prices closer together.

This creation and redemption process happens continuously and helps to keep the ETF price in line with the NAV.

You can find more information in the BlackRock white paper on ETF primary trading and the role of authorized participants (opens in new tab). ☐

ETF arbitrage generally keeps an ETF's market price close to its NAV, which is reassuring evidence that LOOP is a reasonable axiom to start working from. That said, the mechanism cannot be perfect: if there were never a gap between price and NAV, APs would have no profit incentive to engage in creation and redemption in the first place, a version of the Grossman-Stiglitz paradox for informationally efficient markets. In practice these deviations are usually small, but they can widen substantially when the underlying assets are illiquid or hard to trade, as happened with some bond ETFs during the March 2020 market stress.

One of the most important ETFs out there is the SPDR S&P 500 ETF Trust which seeks to provide investment results that, before expenses, correspond generally to the price and yield performance of the S&P 500 Index. The figure below plots the monthly returns of SPDR (Ticker: SPY) since the ETF was listed in 1993.

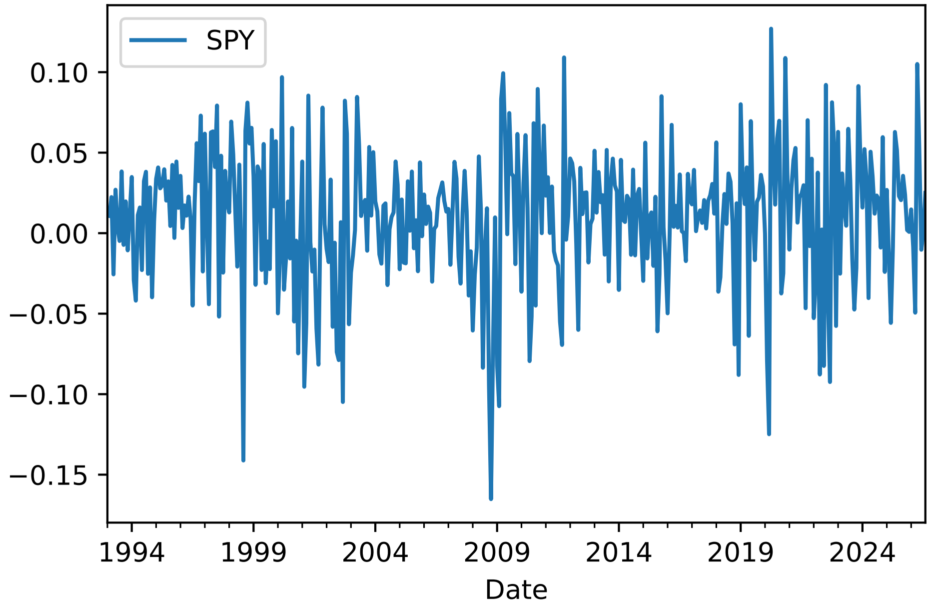

Figure 2: The figure shows the monthly returns of SPDR (SPY) since the ETF was listed in 1993.

Practice Problems

These problems give you a chance to practice the concepts introduced in this chapter. Try to solve each one on your own before expanding the solution.

Solutions to all problems can be found at https://lorenzonaranjo.com/class-materials/investments/.

Problem 1 (Holding-Period Return)**.** You buy a share of a REIT for \$50. Over the year it pays \$2 in dividends, and you sell it a year later for \$53. Compute the capital gain, the income yield, and the HPR.

Problem 2 (Expected Return and Volatility)**.** A stock's HPR over the next year depends on the state of the economy according to the scenario below.

State	Probability	HPR
Boom	0.25	20%

State	Probability	HPR
Normal	0.50	5%
Bust	0.25	-10%

Compute the expected return μ and the volatility σ of the stock.

Problem 3 (Annualizing Monthly Statistics)**.** Using several years of data, you estimate that a stock's average monthly return is 0.9% with a monthly standard deviation of 5.5%. Assuming monthly returns are independent and identically distributed, compute the annualized mean and standard deviation.

Problem 4 (Portfolio Weights and Return)**.** You have \$20,000 to invest. You put \$12,000 in asset A, which returns 8%, and the remaining \$8,000 in asset B, which returns -3%. Compute the portfolio weights w_A and w_B, and the portfolio's HPR.

Problem 5 (Portfolio Scenario Analysis)**.** You invest 60% of your wealth in stock C and 40% in stock D. Their HPRs depend on the state of the economy as follows.

Market	Probability	Stock C	Stock D
Bull	0.4	25%	5%
Normal	0.4	10%	8%
Bear	0.2	-20%	-2%

Compute the portfolio's HPR in each state of the market, and then its expected return and standard deviation.

Problem 6 (The Law of One Price and ETF Arbitrage)**.** Suppose an ETF's market price is currently *below* its Net Asset Value (NAV). Describe the trade an Authorized Participant (AP) would execute to profit from this discrepancy, and explain how this trade affects the ETF's market price and its NAV.

Problem 7 (Holding-Period Return on a Bond)**.** An investor purchased a bond one year ago for \$980. He received \$17 in interest and sold the bond for \$987. What is the holding-period return on his investment?

Problem 8 (Pricing a Risky Portfolio via the Risk Premium)**.** Consider a risky portfolio. The end-of-year cash flow derived from the portfolio will be either \$70,000 or \$200,000 with equal probabilities of 50%. The alternative risk-free investment in T-bills pays 6% per year.

a. If you require a risk premium of 8% so that the discount rate is 6 + 8 = 14%, how much will you be willing to pay for the portfolio?
b. Suppose that the portfolio can be purchased for the amount you found in a. What will be the expected rate of return on the portfolio?
c. Now suppose that you require a risk premium of 12%. What is the price that you will be willing to pay?

Problem 9 (Mean and Standard Deviation of Stock Returns)**.** The stock of company XYZ currently trades at \$100 and just paid a dividend of \$1.8. Suppose that your expectations regarding the stock price and dividends next year are as follows:

State of the Market	Probability	Dividend	Price
Boom	0.3	\$3	\$120
Normal growth	0.5	\$2	\$100
Recession	0.2	\$1	\$80

Compute the mean and standard deviation of the returns for company XYZ.

Problem 10 (Covariance and Correlation of Two Securities)**.** Suppose the economy can only be in one of the following two states: (i) Boom or "good" state and (ii) Recession or "bad" state. Each of the states can occur with an equal probability. At the beginning of a month, you can purchase the following two securities in the market:

- Security 1 is currently trading at \$4. At the end of the month, the security price is expected to increase by \$10 in the good state, and expected to remain unchanged in the bad state.

- Security 2 is currently trading at \$5. At the end of the month, the price of security 2 is expected to remain unchanged in the good state and expected to increase by \$10 in the bad state.

a. Compute the expected returns of securities 1 and 2.
b. Compute the standard deviations of returns for securities 1 and 2.
c. Compute the covariance and the correlation between the returns of the two securities.