Perfect Correlation

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Perfect correlation means one asset's return is an exact linear function of the other's, not approximately, but exactly. That exactness has a striking consequence: whether or not you can diversify away risk becomes a deliberate choice of weights rather than a byproduct of holding several assets. When two assets are perfectly negatively correlated, a long-only combination in the right proportions cancels their fluctuations completely. When they are perfectly positively correlated, no long-only combination helps at all, but going long one asset and short the other in the right proportion still drives risk exactly to zero.

This ability to build an exactly riskless portfolio out of risky assets is not just a mathematical curiosity. Because a riskless portfolio cannot earn anything other than the risk-free rate without creating an arbitrage opportunity, perfectly correlated assets pin down a **synthetic risk-free rate** implied by their prices. This same no-arbitrage logic, applied to an asset and an option written on it, is the foundation on which Black and Scholes (1973) and Merton (1973) built modern derivatives pricing.

The Zero Variance Portfolio

When the correlation between two risky assets A and B is either one or minus one, we say that the assets are **perfectly correlated**. In this case, it is possible to create a portfolio P between the two assets with zero variance.

We saw in the previous note that the variance of a portfolio in which we invest w_A in A and w_B in B is

$$\sigma_P^2 = w_A^2\sigma_A^2 + w_B^2\sigma_B^2 + 2w_Aw_B\sigma_{A,B}. \tag{1}$$

We start first considering the case where $\rho_{A,B} = 1$. In this case, we have that $\sigma_{A,B} = \sigma_A \sigma_B$ and we can write equation (1) as

$$\begin{aligned} \sigma_P^2 &= w_A^2 \sigma_A^2 + w_B^2 \sigma_B^2 + 2 w_A w_B \sigma_A \sigma_B \\ &= (w_A \sigma_A + w_B \sigma_B)^2. \end{aligned} \tag{2}$$

Since $w_A = 1 - w_B$, equation (2) can be made equal to zero by using

$$w_B = \frac{\sigma_A}{\sigma_A - \sigma_B}.$$

Similarly, if $\rho_{A,B} = -1$, equation (1) implies that

$$\sigma_P^2 = (w_A \sigma_A - w_B \sigma_B)^2. \tag{3}$$

Substituting $w_A = 1 - w_B$, the previous expression can be made equal to zero if we pick

$$w_B = \frac{\sigma_A}{\sigma_A + \sigma_B}.$$

Therefore, we conclude that by picking

$$w_B = \begin{cases} \dfrac{\sigma_A}{\sigma_A - \sigma_B} & \text{if } \rho_{A,B} = 1, \\ \dfrac{\sigma_A}{\sigma_A + \sigma_B} & \text{if } \rho_{A,B} = -1, \end{cases} \tag{4}$$

and $w_A = 1 - w_B$ we can make the variance of the portfolio equal to zero.

The portfolio characterized by w_A and w_B in (4) is the global minimum variance portfolio that achieves a variance equal to zero when the two assets are perfectly correlated. In probability, a random variable with zero variance must be constant. Thus, in the absence of arbitrage opportunities, the return of this portfolio must be equal to the risk-free rate. If not, you could borrow at a cheaper rate and invest at a higher rate without risk, generating arbitrarily large profits for free. Such a riskless profit would only last briefly in competitive financial markets.

Thus, we must have that

$$r_f = \mathrm{E}(r_P) = w_A r_A + w_B r_B, \tag{5}$$

where w_B is determined by equation (4) and $w_A = 1 - w_B$. The expression in (5) says that in the absence of arbitrage opportunities, it is possible to create your own risk-free asset if you can trade two perfectly correlated assets.

These assets typically do not exist as such in financial markets, but financial institutions can create them. For example, a *call option* is a contract that gives its purchaser the right but not the obligation to *purchase* an asset at a specific date in the future for a price agreed upon today. Call options exhibit positive perfect correlation with their underlying asset over short intervals. Thus, combining the underlying asset with a call option written on it can create an overnight risk-free asset. This remarkable insight allowed Black and Scholes (1973) and Merton (1973) to derive a formula for pricing derivatives!

A *put option* gives an example of an asset exhibiting a perfect negative correlation with an asset over short periods of time. A put gives its purchaser the right but not the obligation to *sell* an asset at a specific date in the future for a price agreed upon today. Again, combining the underlying asset with a put option written on it can create a risk-free asset. The idea of synthesizing a risk-free asset out of perfectly correlated risky assets has spawned a gigantic industry of derivatives products.

Example 1 (Perfect Positive Correlation)**.** Suppose you have two risky assets A and B such that $\mu_A = 14\%$, $\mu_B = 19\%$, $\sigma_A = 20\%$, $\sigma_B = 30\%$, and $\rho_{A,B} = 1$.

We can use the expression for w_B defined in (4) to compute the implied risk-free rate. Thus, the portfolio defined by

$$w_B = \frac{0.20}{0.20 - 0.30} = -2,$$

and $w_A = 1 - (-2) = 3$ has zero variance. The implied risk-free rate is

$$r_f = 3 \times 0.14 - 2 \times 0.19 = 4\%.$$

We can verify that the variance of the portfolio is indeed zero,

$$\sigma_P = 3 \times 0.2 - 2 \times 0.3 = 0.$$

□

Example 2 (Perfect Negative Correlation). Suppose you have two risky assets A and B such that $\mu_A = 13\%$, $\mu_B = -2\%$, $\sigma_A = 20\%$, $\sigma_B = 10\%$, and $\rho_{A,B} = -1$.

We can use the expression for w_B defined in (4) to compute the implied risk-free rate. The portfolio characterized by

$$w_B = \frac{0.20}{0.20 + 0.10} = 2/3,$$

and $w_A = 1/3$, has zero variance. The implied risk-free rate is

$$r_f = 1/3 \times 0.13 + 2/3 \times (-0.02) = 3\%.$$

The variance of the portfolio is indeed zero since

$$\sigma_P = 1/3 \times 0.2 - 2/3 \times 0.1 = 0.$$

□

Equation (5) can be written as $w_A R_A + w_B R_B = 0$, or

$$R_B = \pm \frac{\sigma_B}{\sigma_A} R_A, \tag{6}$$

where capital letters denote excess returns over the risk-free rate. The sign in equation (6) is the same as the correlation coefficient between the two assets. Thus, if two risky assets are perfectly correlated, their excess returns over the risk free rate must be proportional.

Creating Perfectly Correlated Assets

The converse of the previous statement is also true. Suppose you start with a risky asset A and you create a new asset B with proportions w in A and $1 - w$ in the risk-free asset. The returns of B are described by

$$r_B = (1 - w) r_f + w r_A = r_f + w(r_A - r_f),$$

or

$$R_B = wR_A. \tag{7}$$

We show now that the excess returns of A and B are perfectly correlated with correlation 1 or -1, the sign of the correlation depending on the sign of w.[1]

To see this, we can compute

$$\begin{aligned} \text{Cov}(R_A, R_B) &= \text{Cov}(R_A, wR_A) \\ &= w\sigma_A^2. \end{aligned}$$

Thus, the correlation between A and B is

$$\rho_{A,B} = \frac{\text{Cov}(R_A, R_B)}{\sigma_A \sigma_B} = \frac{w\sigma_A^2}{\sigma_A |w| \sigma_A} = \begin{cases} 1 & \text{if } w > 0, \\ -1 & \text{if } w < 0. \end{cases}$$

By combining any risky asset with the risk-free rate we obtain a whole family of portfolios that are perfectly correlated, either positively or negatively, with each other.

The Investment Opportunity Set

We just saw that we can make perfectly correlated assets out of a risky asset and the risk-free rate. Take a risky asset A with expected return equal to μ_A and standard deviation equal to σ_A.

$$\begin{aligned} \mu_P &= (1 - w) r_f + w\mu_A, \\ \sigma_P &= |w| \sigma_A. \end{aligned}$$

[1] Remember that a negative w means that you short A to invest more in the risk-free asset.

Combining the previous two expressions, we find that the investment opportunity set of combining A and the risk-free asset is given by:

$$\mu_P = \begin{cases} r_f + \left(\frac{\mu_A - r_f}{\sigma_A}\right)\sigma_P & \text{if } w > 0, \\ r_f - \left(\frac{\mu_A - r_f}{\sigma_A}\right)\sigma_P & \text{if } w < 0. \end{cases} \tag{8}$$

The previous expression describes two lines that have the same intercept equal to the risk-free rate. The slope coefficient of the lines is equal to plus/minus the Sharpe ratio of asset A, depending on whether you invest or borrow asset A. This is the same capital allocation line from before, now recovered as a special case of combining two perfectly correlated assets. Figure 1 plots the investment opportunity set generated by asset A and the risk-free asset.

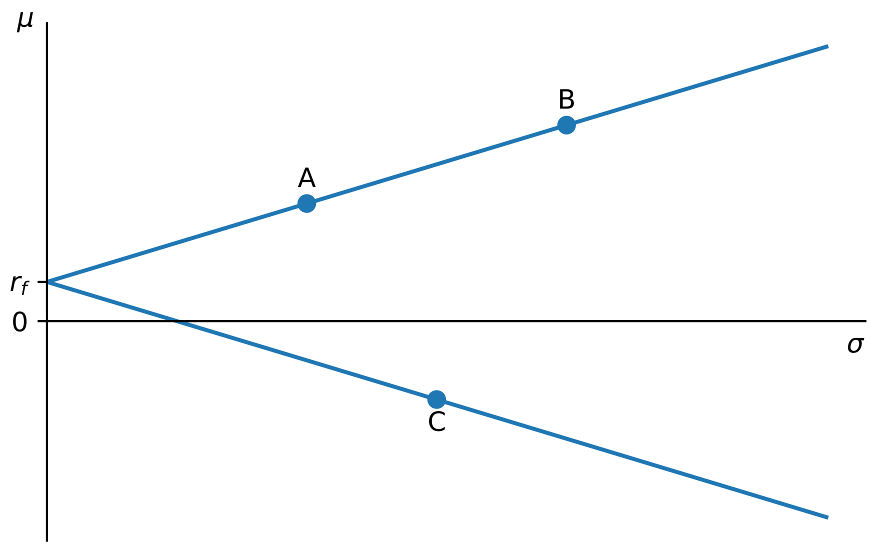

Figure 1: The figure shows the investment opportunity set of combining a risky asset A and the risk-free asset.

In the figure, the point B denotes a portfolio between A and the risk-free asset where $R_B = wR_A$ and $w > 1$. Thus, assets A and B are perfectly positively correlated. If there was no risk-free asset, both risky assets could be combined together to create a risk-free asset.

Example 3. Suppose you have two risky assets A and B such that $\mu_A = 15\%$, $\mu_B = 25\%$, $\sigma_A = 25\%$, $\sigma_B = 50\%$, and $\rho_{A,B} = 1$.

This would be the situation described in Figure 1. The slope coefficient of the line created by A and B is

$$SR = \frac{\mu_B - \mu_A}{\sigma_B - \sigma_A} = \frac{0.25 - 0.15}{0.50 - 0.25} = 0.40.$$

The line between A and B is described by

$$\mu = r_f + 0.40\sigma,$$

where r_f is the risk-free rate and represents the line intercept with the y-axis. The equation should be valid for both A and B, so we can pick either to compute the implied risk-free rate. If we use asset A we have that

$$0.15 = r_f + 0.40 \times 0.25,$$

or

$$r_f = 0.15 - 0.40 \times 0.25 = 5\%.$$

□

Practice Problems

These problems give you a chance to practice the concepts introduced in this chapter. Try to solve each one on your own before expanding the solution.

Solutions to all problems can be found at https://lorenzonaranjo.com/class-materials/investments/.

Problem 1 (A Zero-Variance Portfolio with Perfect Positive Correlation)**.** A pension fund's compliance officer is reviewing two structured notes, M and N, pitched by different investment banks. Both notes are written on the same underlying index, so the officer suspects their returns are perfectly positively correlated, $\rho_{M,N} = 1$. Note M offers expected return $\mu_M = 16\%$ with volatility $\sigma_M = 18\%$, while note N offers expected return $\mu_N = 22\%$ with volatility $\sigma_N = 27\%$.

Before recommending either product to the fund's board, the officer wants to check whether the two notes are consistently priced. If they are not, some combination of a long position in one and a short position in the other could lock in a riskless return that differs from the market's actual risk-free rate, exposing an arbitrage opportunity the banks have overlooked. Find the weights of the zero-variance portfolio combining M and N, and use them to compute the implied risk-free rate.

Problem 2 (A Zero-Variance Portfolio with Perfect Negative Correlation)**.** A hedge fund analyst is studying two risky assets, Y and Z, that his fund's risk model flags as perfectly negatively correlated, $\rho_{Y,Z} = -1$. Asset Y has expected return $\mu_Y = 11\%$ and volatility $\sigma_Y = 24\%$, while asset Z has expected return $\mu_Z = 1\%$ and volatility $\sigma_Z = 16\%$.

Since the two assets move in exactly opposite directions, the analyst suspects he can combine them into a portfolio with zero variance, effectively manufacturing a synthetic risk-free asset. Find the weights of this zero-variance portfolio, and use them to compute the implied risk-free rate his fund could lock in.

Problem 3 (Creating Perfectly Correlated Assets Through Leverage)**.** A trader at a proprietary trading desk covers a single risky stock, A, with expected return $\mu_A = 18\%$ and volatility $\sigma_A = 22\%$. The desk can borrow or lend freely at the risk-free rate of 4%.

One morning, the trader's manager asks her to stress-test two alternative strategies built entirely out of A and the risk-free asset, each represented by a portfolio B in which weight w is placed in A and $1 - w$ in the risk-free asset. The manager wants to know how each strategy's risk and return compare to the original stock, and whether the two are still driven by the same underlying source of risk. Find μ_B, σ_B, and the correlation between the excess returns of A and B for each strategy.

a. First, the trader considers levering up: borrowing at the risk-free rate to invest $w = 1.5$ in A.
b. Next, she considers the opposite bet: shorting the stock with $w = -0.8$ and parking the proceeds in the risk-free asset.

Problem 4 (Recovering the Risk-Free Rate from Two Correlated Assets)**.** An analyst is studying two mutual funds, P and Q, that follow such closely related strategies that their returns are perfectly positively correlated, $\rho_{P,Q} = 1$. Fund P has historically delivered an expected return $\mu_P = 12\%$ with volatility $\sigma_P = 15\%$, while the more aggressive fund Q has delivered $\mu_Q = 21\%$ with volatility $\sigma_Q = 40\%$. The analyst does not have direct access to the risk-free rate that prevailed when these figures were estimated, but knows that since P and Q are perfectly correlated, they must lie on the same straight line through the risk-free rate in (σ, μ)-space.

Using only the data on P and Q, find the slope of this line and use it to back out the implied risk-free rate.

Problem 5 (Detecting an Arbitrage Opportunity). A junior trader notices that two risky assets, X and Y, have returns that move in exactly opposite directions: whenever one goes up, the other goes down by a proportional amount, so that $\rho_{X,Y} = -1$. Asset X has an expected return $\mu_X = 10\%$ and a volatility $\sigma_X = 12\%$, while asset Y has an expected return $\mu_Y = 1\%$ and a volatility $\sigma_Y = 18\%$. Meanwhile, the trading desk can borrow or lend at the prevailing risk-free rate, currently quoted at 5%.

The trader suspects that combining X and Y in the right proportions might let the desk lock in a riskless profit at no cost. Determine whether such an arbitrage opportunity actually exists. If it does, explain precisely how the desk should trade to exploit it, and compute the riskless profit earned per dollar borrowed.

References

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