Optimal Capital Allocation

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If passive investors possess the same information, they should agree on which combination of risky assets provides the best trade off between risk and return. Nevertheless, some investors may think this optimal portfolio of risky assets carries too much risk. Others may think the opposite.

It is possible to reduce or increase the risk of a portfolio by investing or borrowing a risk-free asset. The **capital allocation** decision is then about how much to invest in this well-diversified portfolio of risky investments and how much to allocate to a risk-free asset.

The solution to the capital allocation problem has two dimensions: we must model what is available to invest, and we must understand what investors want.

The first point requires us to determine the **investment opportunity set**. For example, we would love to be able to invest our savings in a risk-free deposit account that yields 20% per year. Unfortunately, such an account does not exist. Combining a risky asset with the risk-free rate generates an investment set called the **capital allocation line**.

The second point has to do with how investors feel when taking risks. Investors dislike risk but like returns. In finance and economics, we capture these two opposite effects using a **utility function**.

The Capital Allocation Line

We will now analyze the investment opportunity set generated by a risky asset Q and a risk-free asset. We denote by r_f the risk-free rate of return. The expected return of the risky asset is denoted by μ_Q whereas the standard deviation or volatility of the risky asset's returns is denoted by σ_Q.

A portfolio P that invests w in Q and $1-w$ in the risk-free asset has the following expected return and volatility:

$$\begin{aligned}\mu_P &= (1-w)r_f + w\mu_Q,\\ \sigma_P &= |w|\sigma_Q. \end{aligned} \tag{1}$$

If we only consider portfolios in which we invest in the risky asset, i.e. $w \geq 0$, we can combine both equations to get

$$\mu_P = r_f + \left(\frac{\mu_Q - r_f}{\sigma_Q}\right)\sigma_P.$$

The Sharpe ratio of the risky-asset is defined as:

$$SR = \frac{\mu_Q - r_f}{\sigma_Q}. \tag{2}$$

Therefore, if the x-axis is represented by σ and the y-axis is represented by μ, the expected return of any portfolio formed by combining the risk-free and the risky asset is given by a line with intercept r_f and slope coefficient *SR*:

$$\mu = r_f + SR \times \sigma.$$

This line is called the *Capital Allocation Line* of Q or just CAL(Q). When the risky-asset is the market portfolio the CAL(Q) is called the *Capital Market Line* (CML).

The CAL(Q) transforms risk into expected returns. It can be seen as a *production function*, just like the one we used in the Fisher model, where now the input is *risk* and the output is *expected return*. Thus, the Sharpe ratio of Q is the *marginal rate of transformation* (MRT) of risk into expected return. When Q is the market portfolio, this production function is exactly the CML.

Example 1. Suppose that $r_f = 5\%$, $\mu_Q = 12\%$ and $\sigma_Q = 20\%$. The Sharpe ratio of Q is

$$SR = \frac{0.12 - 0.05}{0.20} = 0.35.$$

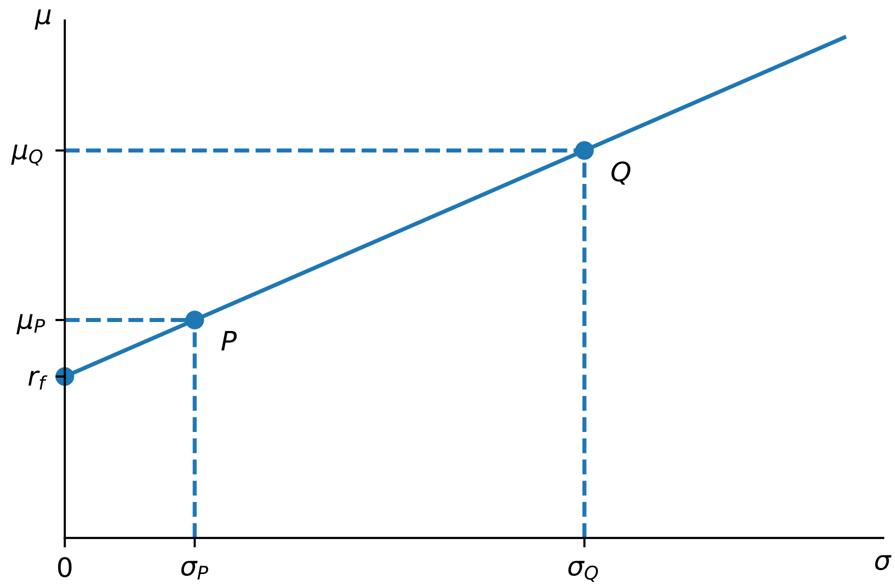


Figure 1: The figure shows the capital allocation line of Q.

Suppose that you want a portfolio P on the CAL(Q) but with 5% volatility. If w denotes the weight in Q, this means that

$$0.05 = w \times 0.20,$$

or $w = 25\%$. Thus, a portfolio that invests 25% in Q and 75% in the risk-free asset has 5% volatility. The expected return of this portfolio is:

$$\mu_P = 0.75 \times 0.05 + 0.25 \times 0.12 = 6.75\%.$$

The position of P in the CAL(Q) is illustrated in Figure 1. □

Example 2. Highland Capital Advisors (HCA) uses the capital market line to make asset allocation recommendations. HCA derives the following forecasts: an expected return on the market portfolio of $\mu_M = 12\%$, a standard deviation on the market portfolio of $\sigma_M = 20\%$, and a risk-free rate of $r_f = 5\%$.

Elena Cross seeks HCA's advice for a portfolio asset allocation. Cross informs HCA that she wants the standard deviation of the portfolio to equal half of the standard deviation of the market portfolio. Using the capital market line, what expected return can HCA provide subject to Cross's risk constraint?

Since the risky asset here is the market portfolio, the relevant investment opportunity set is the CML, whose slope is the Sharpe ratio already computed in Example 1, $SR = 0.35$. Cross's risk constraint requires

$$\sigma_P = 0.5 \times \sigma_M = 10.0\%.$$

Plugging into the CML equation gives the expected return HCA can offer:

$$\mu_P = r_f + SR \times \sigma_P = 0.05 + 0.35 \times 0.1 = 8.5\%.$$

This portfolio invests $w = \sigma_P/\sigma_M = 50.0\%$ in the market portfolio and the remaining 50.0% in the risk-free asset. □

Investor's Utility

Investors seek to get the maximum return for the minimum risk. A standard way in economics to capture this trade-off is by using a *utility function*. The idea of introducing a utility function is to be able to rank different combinations of risk vs. return.

Recall from Utility Theory Under Uncertainty that a portfolio's return is a *proportional* gamble on wealth, not a dollar gamble, and that for a small proportional gamble with mean μ and variance σ^2, the certainty-equivalent return of a risk-averse investor is approximately $\mu - \frac{1}{2}RRA \cdot \sigma^2$, where *RRA* is the local coefficient of *relative* risk-aversion evaluated at the investor's wealth. A simple and convenient way to model an investor's preferences over portfolios is to promote this local approximation to a utility function defined over any (μ, σ) pair, i.e.,

$$U(\mu, \sigma) = \mu - \frac{1}{2}A\sigma^2. \tag{3}$$

Here A plays the same role as *RRA* did before: it denotes how sensitive is a particular investor to risk measured here by σ^2. A higher value for A reduces the utility for the same level of risk. We call A the coefficient of risk-aversion. It is common in applications to use values for A between 1 and 4.[1]

This means that there are several pairs (μ, σ) that provide the same utility, i.e. for a given U and σ, we can always find a μ such that:

$$\mu = U + \frac{1}{2} A\sigma^2.$$

These functions are called *indifference curves* since an investor with risk-aversion coefficient A is indifferent among any of these combinations of μ and σ.

Let's fix $A = 3$ and U to be either 2%, 6% or 10%. We can now plot the corresponding indifference curves.

The curves in the graph represent all combinations of (μ, σ) that provide the same utility, i.e. the investor is *indifferent* among these choices of risk and return. Indifference curves that provide higher utility are always above indifference curves that provide lower utility.

Each indifference curve can be characterized by its *certainty equivalent*, which represents the expected return that would provide the same level of utility with no risk, that is when $\sigma = 0$. The utility level can therefore be interpreted as the certainty equivalent of a particular portfolio.

[1] Unlike the local approximation above, which holds only approximately and only for small gambles, (3) can be obtained *exactly* from expected utility under exponential (CARA) utility, $u(W) = -e^{-aW}$ as in Utility Theory Under Uncertainty, together with normally distributed portfolio returns. Let W_0 be initial wealth and $\tilde{r} \sim N(\mu, \sigma^2)$ the portfolio's return, so that terminal wealth is $\tilde{W} = W_0(1 + \tilde{r})$. By the moment-generating function of the normal distribution, $\mathrm{E}\left(e^{-aW_0\tilde{r}}\right) = e^{-aW_0\mu + \frac{1}{2}(aW_0)^2\sigma^2}$, so that

$$\begin{aligned} \mathrm{E}(u(\tilde{W})) &= -e^{-aW_0}\,\mathrm{E}\left(e^{-aW_0\tilde{r}}\right) \\ &= -e^{-aW_0} \cdot e^{-aW_0\mu + \frac{1}{2}(aW_0)^2\sigma^2} \\ &= -\exp\left[-aW_0\left((1+\mu) - \frac{1}{2}(aW_0)\sigma^2\right)\right]. \end{aligned}$$

Since $-e^{-x}$ is strictly increasing in x, ranking portfolios by expected utility is the same as ranking them by $\mu - \frac{1}{2}(aW_0)\sigma^2$, which is exactly (3) with $A = aW_0$, the coefficient of relative risk-aversion *RRA* evaluated at W_0, consistent with the local argument above.

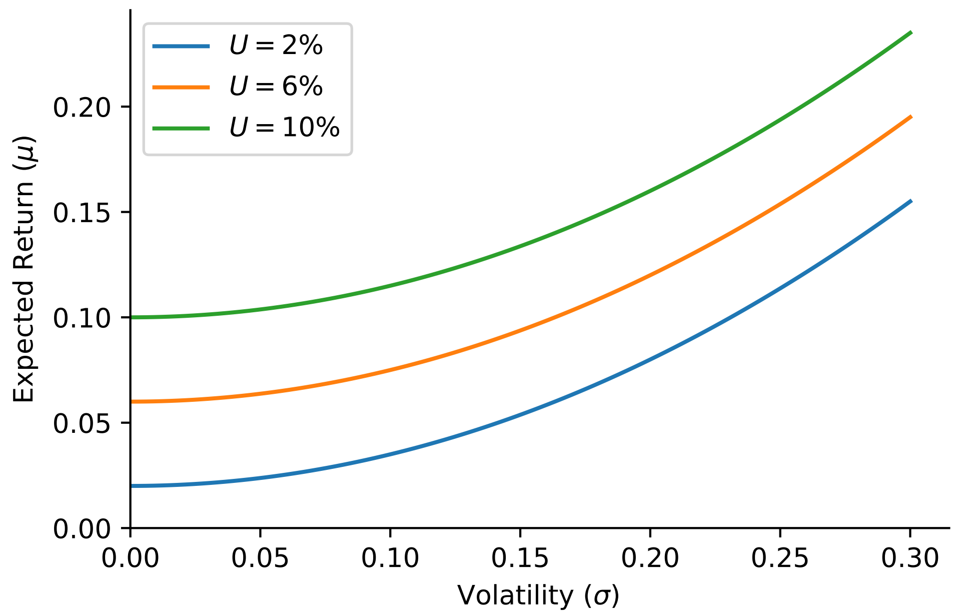


Figure 2: The figure shows indifference curves for different levels of utility.

Maximizing Utility

Optimal portfolio choice is about maximizing utility given the constraints imposed by the investment opportunity set. For a given w that determines the weight in the risk asset Q, the investment opportunity set is characterized by:

$$\begin{aligned} \mu &= (1-w)r_f + w\mu_Q, \\ \sigma^2 &= w^2\sigma_Q^2. \end{aligned} \tag{4}$$

The utility of investing w in Q and the rest in the risk-free asset :

$$\begin{aligned} U &= \mu - \frac{1}{2}A\sigma^2 \\ &= (1-w)r_f + w\mu_Q - \frac{1}{2}Aw^2\sigma_Q^2. \end{aligned}$$

The first-order condition (FOC) is:

$$\frac{dU}{dw} = (\mu_Q - r_f) - Aw\sigma_Q^2 = 0.$$

implying that the optimal w^* is given by:

$$w^* = \frac{\mu_Q - r_f}{A\sigma_Q^2}.$$

The previous expression shows that the amount allocated to the risky asset is smaller if the risk aversion or if its variance are larger. We can see that in terms of allocation to the risky asset, the investor's risk aversion and the variance of the asset play the same role and are indistinguishable. Certainly, we can also see that the amount allocated to the risky asset increases with its expected return, but decreases with the risk-free rate.[2]

Knowing w^* tells us how much to invest in the risky asset and therefore how much to invest in the risk-free asset. The resulting expected return and standard deviation of the optimal portfolio are given by:

$$\begin{aligned}\mu^* &= (1 - w^*)r_f + w^*\mu_Q,\\ \sigma^* &= w^*\sigma_Q.\end{aligned}$$

Example 3. Consider an agent with a risk-aversion coefficient equal to 3. If $r_f = 5\%$, $\mu_Q = 12\%$ and $\sigma_Q = 20\%$ as in Example 1, we have that

$$w^* = \frac{0.12 - 0.05}{3 \times 0.20^2} = 58.33\%.$$

Therefore, the portfolio that maximizes the utility for the investor consists of investing 41.67% in the risk-free asset and 58.33% in the risky asset. The expected return and volatility of this portfolio are

$$\begin{aligned}\mu^* &= (1 - w^*) \times 0.05 + w^* \times 0.12 = 9.08\%,\\ \sigma^* &= w^* \times 0.20 = 11.67\%.\end{aligned}$$

□

[2] We will see later that if Q is the market portfolio, in equilibrium we must have that $w^* = 1$. Therefore, an increase in σ_M^2 or A will translate in an increase of μ_M, i.e. a decrease in prices.

The optimal portfolio has the following interpretation. The investor wants to maximize utility, i.e., to achieve the highest level of utility possible. The point where the indifference curve is tangent to the CAL(Q) determines the optimal portfolio. At this point, the *marginal rate of substitution* between risk and return equals the *marginal rate of transformation* between risk and return — the same tangency condition that pinned down the optimal choice between consumption today and tomorrow in the Fisher model, only now applied to risk and return instead of C_0 and C_1.

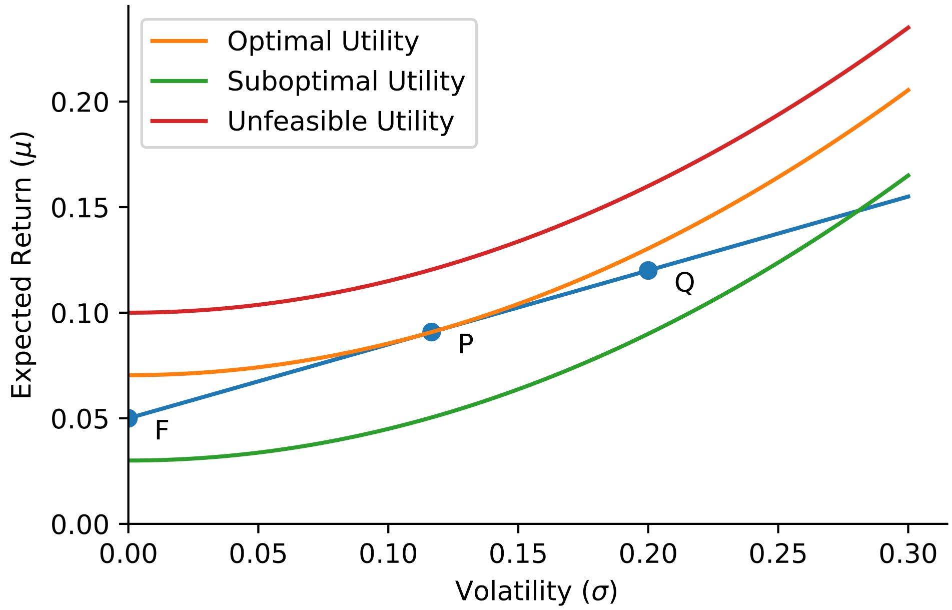


Figure 3: The figure shows that optimal portfolio choice occurs where the marginal rate of substituion equals the marginal rate of transformation between risk and return.

Practice Problems

These problems give you a chance to practice the concepts introduced in this chapter. Try to solve each one on your own before expanding the solution.

Solutions to all problems can be found at https://lorenzonaranjo.com/class-materials/investments/.

Problem 1 (Sharpe Ratio and Portfolio Weight)**.** A brokerage forecasts a risk-free rate of $r_f =$ 3%, an expected return on the market portfolio of $\mu_M = 11\%$, and a market volatility of $\sigma_M =$

18%. Compute the Sharpe ratio of the market portfolio. If a client wants a portfolio on the CML with 9% volatility, what weight should she hold in the market portfolio, and what is the expected return of this portfolio?

Problem 2 (Targeting an Expected Return on the CML). Using forecasts of $r_f = 4\%, \mu_M = 10\%$, and $\sigma_M = 16\%$, a client asks for a portfolio on the CML with an expected return of 8.5%. What volatility does this portfolio have, and what weight does the client hold in the market portfolio?

Problem 3 (Computing Mean-Variance Utility). An investor with risk-aversion coefficient $A = 4$ is considering a portfolio with expected return $\mu = 9\%$ and volatility $\sigma = 15\%$. Compute her utility level, and interpret it as a certainty equivalent.

Problem 4 (Certainty Equivalents and Points on an Indifference Curve). An investor with risk-aversion coefficient $A = 2$ has a certainty equivalent of $CE = 5\%$ for her current portfolio, i.e., she is indifferent between holding it and receiving a sure return of 5%. What expected return μ must an alternative, riskier portfolio with $\sigma = 10\%$ offer so that she is just as well off holding it instead?

Problem 5 (Optimal Capital Allocation). An investor with risk-aversion coefficient $A = 4$ faces $r_f = 5\%, \mu_Q = 13\%$, and $\sigma_Q = 20\%$. Find the optimal weight w^* in the risky asset, and the resulting expected return and volatility of the optimal portfolio.

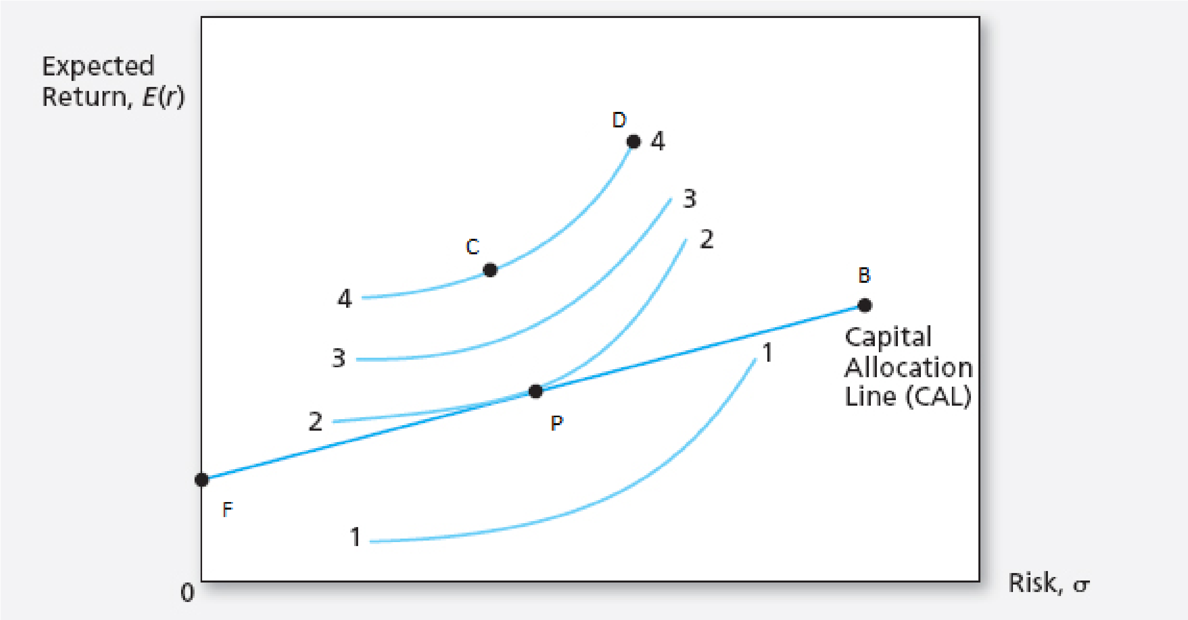
Problem 6 (Comparative Statics: Risk Aversion and Allocation). Two investors share the same forecasts, $r_f = 5\%, \mu_Q = 13\%$, and $\sigma_Q = 20\%$, but investor A has risk-aversion coefficient $A = 2$ while investor B has $A = 8$. Without recomputing everything from scratch, use the formula for w^* to predict how the two investors' allocations to the risky asset should compare. Then verify your prediction by computing w^* for each investor.

Problem 7 (Optimal Allocation for a Client's Portfolio). Assume that you manage a risky portfolio with an expected rate of return of 18% and a standard deviation of 28%. The T-bill rate is 8%. Your client's degree of risk aversion is $A = 3.5$, assuming a utility function $U = \mu - 0.5A\sigma^2$.

a. What proportion of the total investment should be invested in your fund?

b. What is the expected return on your client's optimized portfolio?
c. What is the standard deviation on your client's optimized portfolio?

Problem 8 (Indifference Curves and the Optimal Tangency Portfolio). Consider the following graph that plots four indifference curves of an investor with utility score given by $U = \mu - 0.5A\sigma^2$. The indifference curves are labeled 1 through 4 in increasing order of utility, together with the capital allocation line (CAL) running from the risk-free asset F to the risky asset B. Curve 2 is tangent to the CAL at point P whereas curve 1 crosses the CAL at two points; curves 3 and 4 lie entirely above the CAL and never touch it.



Currently, the investor can only invest in the risk-free asset denoted by F, and a risky asset denoted by B. You know that $\mu_B = 11\%$, $\sigma_B = 15\%$, and that the risk-free asset yields 5%.

a. Which indifference curve (1, 2, 3, or 4) represents the greatest level of utility that can be achieved by the investor? Why?
b. For which value of the risk-aversion coefficient would the optimal portfolio chosen by the investor be composed of 50% in the risky asset?

Problem 9 (Client Portfolio Weights and the Sharpe Ratio). Assume that you manage a risky portfolio with an expected rate of return of 18% and a standard deviation of 28%. The T-bill rate is 8%. Your client chooses to invest 70% of a portfolio in your fund and 30% in a T-bill money market fund.

a. What is the expected value and standard deviation of the rate of return on his portfolio?

b. Suppose that your risky portfolio includes the following investments in the given proportions:

Asset	Proportions
Stock A	25%
Stock B	32%
Stock C	43%

What are the investment proportions of your client's overall portfolio, including the position in T-bills?

c. What is the reward-to-volatility ratio, i.e. the Sharpe ratio, of your risky portfolio and your client's portfolio?

References