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# Class 6

## Perfectly Correlated Assets

Class 5 showed that the cross term in the portfolio variance can remove risk, and that how much it removes depends on the weights given the correlation. The extreme case is **perfect correlation**, $\rho_{AB} = \pm 1$, where one asset's return is an exact linear function of the other's. The portfolio variance then becomes a perfect square,

$$\sigma_P^2 = (w_A \sigma_A \pm w_B \sigma_B)^2, \tag{1}$$

with the same sign as the correlation, so weights that set the term in parentheses to zero remove all risk. Using $w_A = 1 - w_B$, the **zero-variance portfolio** is

$$w_B = \begin{cases} \dfrac{\sigma_A}{\sigma_A - \sigma_B} & \text{if } \rho_{AB} = 1, \\ \dfrac{\sigma_A}{\sigma_A + \sigma_B} & \text{if } \rho_{AB} = -1. \end{cases} \tag{2}$$

With $\rho_{AB} = -1$ both weights are positive, and a long position in each asset offsets their fluctuations. With $\rho_{AB} = 1$ one weight is negative: going long the less volatile asset and short the more volatile one, in the right proportion, cancels the risk exactly.

## A Synthetic Risk-Free Rate

A portfolio with zero variance earns the same return in every state. If markets offer no arbitrage opportunities, that return must equal the risk-free rate,

$$r_f = w_A\mu_A + w_B\mu_B, \tag{3}$$

with the weights from (2). If the zero-variance portfolio paid more than $r_f$, you could borrow at $r_f$ and invest in it, earning a riskless profit with no money down. If it paid less, you would short it and lend at $r_f$. Traders would scale either trade up until prices adjusted, so perfectly correlated assets pin down a **synthetic risk-free rate** implied by their expected returns.

Writing $R_i = r_i - r_f$ for excess returns, (3) says that $w_A R_A + w_B R_B = 0$ in every state, or

$$R_B = \rho_{AB}\frac{\sigma_B}{\sigma_A}R_A. \tag{4}$$

The excess returns of perfectly correlated assets are proportional. Taking expectations, $(\mu_B - r_f)/\sigma_B = \rho_{AB}(\mu_A - r_f)/\sigma_A$. With $\rho_{AB} = 1$ the two assets have the same Sharpe ratio and lie on the same line through $r_f$. With $\rho_{AB} = -1$ their Sharpe ratios are equal in size and opposite in sign, so they lie on two lines that meet at $r_f$, one rising and one falling. Figure 1 draws both cases. In each, the zero-variance portfolio is the point where the opportunity set touches the vertical axis.

## Building Perfect Correlation

Perfectly correlated pairs are not something you find among traded stocks, but they are easy to build. Combining any risky asset $A$ with the risk-free asset gives a portfolio $B$ with $r_B = r_f + w(r_A - r_f)$, that is,

$$R_B = wR_A, \tag{5}$$

so $\rho_{AB} = 1$ if $w > 0$ and $\rho_{AB} = -1$ if $w < 0$. A levered position in $A$ is perfectly positively correlated with $A$, and a short position in $A$ with the proceeds invested at $r_f$ is perfectly negatively correlated with it. The set of all such portfolios is the capital allocation line of

Class 4 together with its lower branch: two rays through $r_f$ with slopes equal to plus and minus the Sharpe ratio of $A$, exactly the shape in Figure 1.

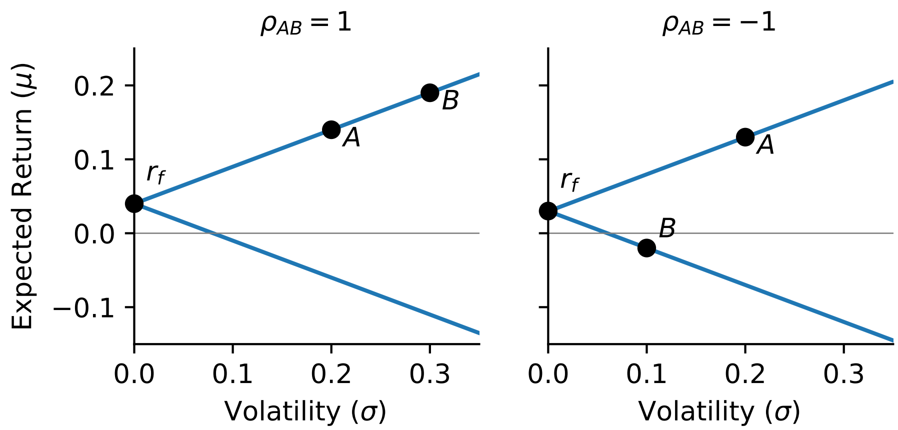


**Figure 1:** Opportunity sets of two perfectly correlated assets. The zero-variance portfolio earns an implied risk-free rate of $4\%$ on the left and $3\%$ on the right.

Derivatives deliver the same property over short intervals. A call option moves with its underlying asset and a put moves against it, so over a short horizon a call is perfectly positively correlated with the underlying and a put is perfectly negatively correlated with it. Combining the option with the right amount of the underlying creates an overnight risk-free position, which by the argument above must earn $r_f$. This is the no-arbitrage argument that Black, Scholes, and Merton used in 1973 to price options.

## Practice Problems

1. Suppose that there are many assets in the security market and that the characteristics of assets A and B are given as follows:

| Asset | Expected Return | Standard Deviation |
|---|---|---|
| A | 5% | 20% |
| B | 20% | 30% |

   The correlation between the asset returns is -1. Suppose that it is possible to invest and borrow at the risk-free rate, $r_f$. What must be the value of the risk-free rate?