Class 4

The Capital Allocation Problem

Investors who share the same information should agree on the best portfolio of risky assets, but they need not agree on how much risk to take. The **capital allocation** decision is how much to invest in that risky portfolio Q and how much in a risk-free asset paying r_f. Solving it requires two ingredients: the **investment opportunity set** (what is feasible) and a **utility function** (what the investor wants).

The Capital Allocation Line

A portfolio P that invests w in Q and $1-w$ in the risk-free asset has

$$\begin{aligned}\mu_P &= (1-w)r_f + w\mu_Q,\\ \sigma_P &= |w|\sigma_Q.\end{aligned} \tag{1}$$

For $w \geq 0$, eliminating w gives the **capital allocation line** CAL(Q):

$$\mu = r_f + SR \times \sigma, \qquad SR = \frac{\mu_Q - r_f}{\sigma_Q},$$

a straight line with intercept r_f and slope equal to the **Sharpe ratio** of Q, shown in Figure 1. When Q is the market portfolio, this line is the **capital market line** (CML).

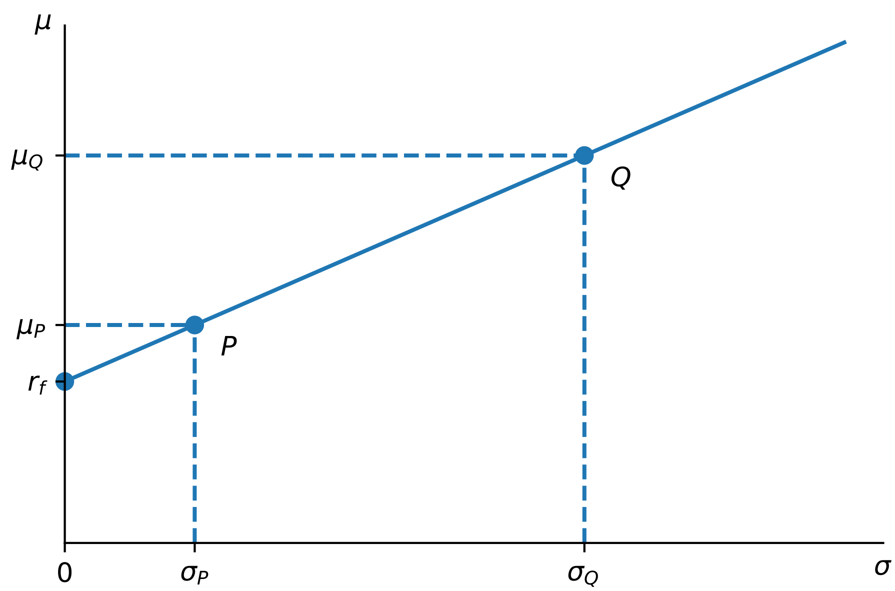

Figure 1: The capital allocation line of Q, for $r_f = 5\%$, $\mu_Q = 12\%$ and $\sigma_Q = 20\%$.

The CAL is a **production function** that converts risk into expected return, so the Sharpe ratio is the **marginal rate of transformation** of risk into expected return. Moving along the CAL by changing w does not change its slope: the Sharpe ratio depends only on the composition of the risky portfolio, not on how much of it you hold.

Mean-Variance Utility

Preferences over (μ, σ) pairs are summarized by

$$U = \mu - \frac{1}{2}A\sigma^2, \tag{2}$$

where A is the **coefficient of risk aversion**, playing the role of relative risk aversion in the local approximation of the certainty equivalent of a small proportional gamble. Typical values are between 1 and 4.

Combinations of (μ, σ) giving the same utility trace an **indifference curve**,

$$\mu = U + \frac{1}{2}A\sigma^2,$$

an upward-sloping parabola whose intercept U is the portfolio's **certainty equivalent**. Higher curves mean higher utility, as Figure 2 shows for $A = 3$.

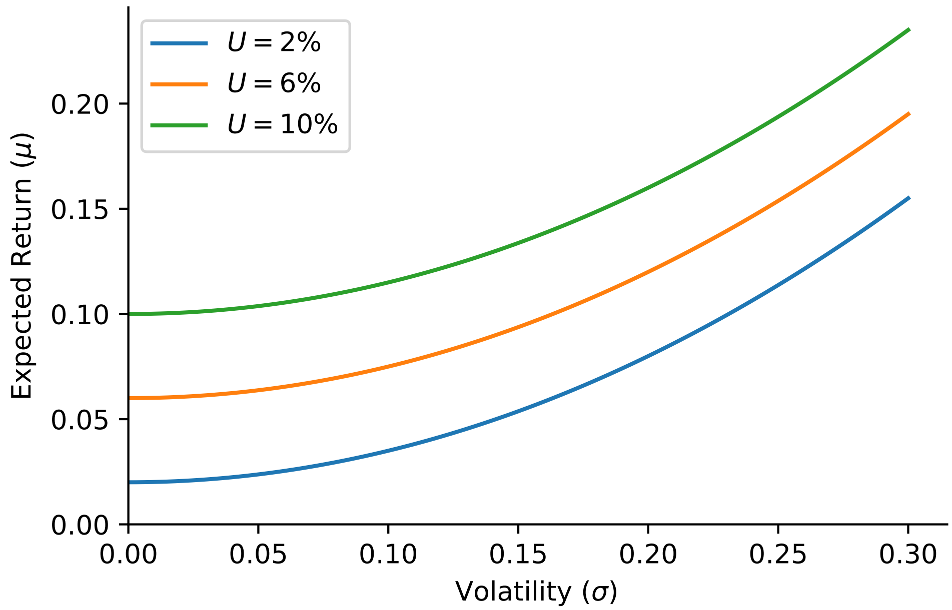

Figure 2: Indifference curves for different levels of utility, with $A = 3$.

Optimal Capital Allocation

Substituting the investment opportunity set into the utility function,

$$U = (1 - w)r_f + w\mu_Q - \frac{1}{2}Aw^2\sigma_Q^2,$$

the first-order condition $(\mu_Q - r_f) - Aw\sigma_Q^2 = 0$ gives

$$w^* = \frac{\mu_Q - r_f}{A\sigma_Q^2},$$

with $\mu^* = (1 - w^*)r_f + w^*\mu_Q$ and $\sigma^* = w^*\sigma_Q$. The allocation to the risky asset rises with its risk premium and falls with risk aversion and variance, which enter symmetrically. Geometrically, the optimum is the point where an indifference curve is *tangent* to the CAL: the marginal rate of substitution between risk and return equals the marginal rate of transformation, as illustrated in Figure 3.

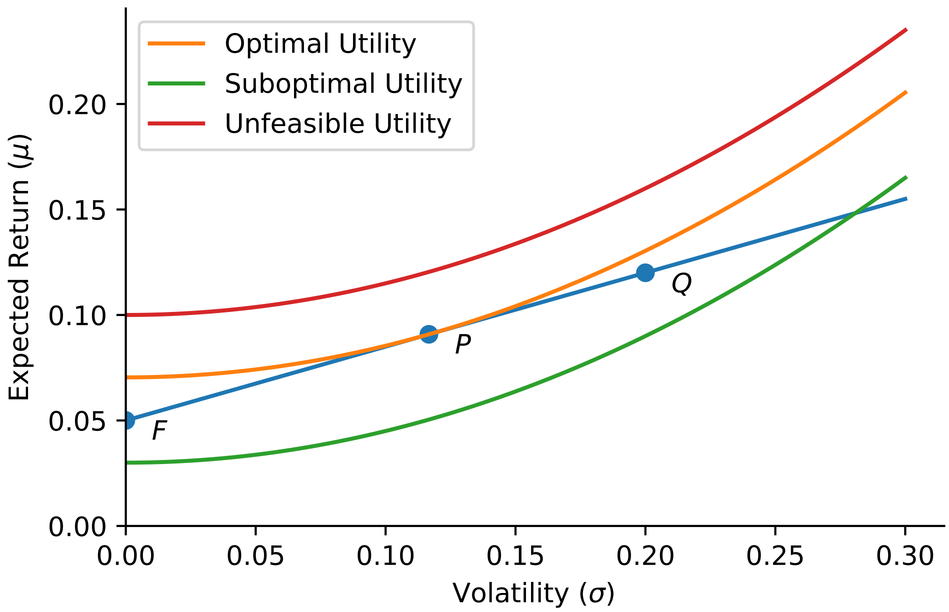

Figure 3: Optimal portfolio choice occurs where the marginal rate of substitution equals the marginal rate of transformation between risk and return.

Practice Problems

1. Investment Management Inc. (IMI) uses the capital market line to make asset allocation recommendations. IMI derives the following forecasts:

   - Expected return on the market portfolio: 12%.
   - Standard deviation on the market portfolio: 20%.
   - Risk-free rate: 5%.

   Samuel Johnson seeks IMI's advice for a portfolio asset allocation. Johnson informs IMI that he wants the standard deviation of the portfolio to equal half of the standard deviation for the market portfolio. Using the capital market line, what expected return can IMI provide subject to Johnson's risk constraint?

2. Suppose that you have \$1 million and the following two opportunities from which to construct a portfolio:

   - Risk-free asset earning 5% per year.
   - Risky asset with expected return of 30% per year and standard deviation of 40%.

   a. If you construct a portfolio with a standard deviation of 30%, what is its expected rate of return?

   b. If your utility is given by $U = \mu - \frac{1}{2}A\sigma^2$, compute the optimal allocation in the risky and risk-free asset, and the expected return and standard deviation of your optimal portfolio if $A = 2$.