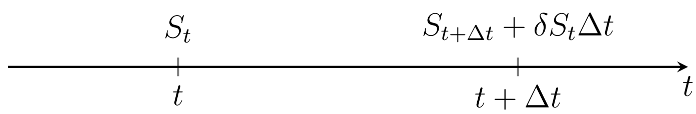
Options on Assets Paying a Dividend Yield

General Framework

The Dividend Yield

It is usually convenient to model dividends as a percentage yield paid over time. We will denote the continuously-compounded dividend yield by δ. The asset S then pays every instant t a dividend of $\delta \Delta t$ shares and therefore $\delta S_t \Delta t$ dollars. Therefore, if you purchase one unit of the asset at time t for S_t, the value of the portfolio at time $t + \Delta t$ will be $S_{t+\Delta t} + \delta S_t \Delta t$.



You will notice that the payment of the dividend at time $t + \Delta t$ is known at time t. Indeed, this payment depends on the value of the stock at time t and not $t + \Delta t$. This is not a mistake or a convenience, but reflects the fact that we want to incorporate in the total return of investing in the stock from time t to $t + \Delta t$ both capital gains and dividends:

$$\underbrace{\frac{S_{t+\Delta t} + \delta S_t \Delta t - S_t}{S_t}}_{\text{Total Return}} = \underbrace{\frac{\Delta S_t}{S_t}}_{\text{Capital Gains}} + \underbrace{\delta \Delta t}_{\text{Dividends}} . \tag{1}$$

In practice, this is the approach used to model options on stock indices and foreign currencies, although some practitioners also use it to model individual stocks as well. We must note that specially for American type options, modelling lump-sum dividends as a continuous yield might induce errors in computing the optimal early-exercise policy. It could also lead to the wrong risk-neutral adjustment if dividends are paid, say, twice per year, and we want to risk-adjust the underlying asset process for the next three months just after a dividend has been paid.

Replicating A Derivative

As we did in the previous chapter where there were no dividends, to price a call or put option we take the point of view of a trading desk that makes the market for such contracts. Their sales team just sold a European option H written on a stock S with maturity T to a client. The stock in this case, though, pays a continuous dividend yield δ.

The traders of the desk will replicate the option by buying (or selling) N_S units of the stock and N_β units of a zero-coupon bond with face value K and maturity T, respectively. The difference in this case compared to the no-dividend stock is that the number of shares bought will grow, so that the trader will have to buy a little less in order to hedge a call option, for example.

If we call V the value of such replicating portfolio, we have that:

$$V_t = N_{S,t}S_t + N_{\beta,t}\beta_t. \tag{2}$$

At time $t+\Delta t$, and because the underlying asset pays dividends, the value of the replicating portfolio is:

$$V_{t+\Delta t} = N_{S,t}(S_{t+\Delta t} + \delta S_t \Delta t) + N_{\beta,t}\beta_{t+\Delta t},$$

which implies that:

$$\Delta V_t = N_{S,t}(\Delta S_t + \delta S_t \Delta t) + N_{\beta,t}\Delta\beta_t.$$

As $\Delta t \to 0$, we have that:

$$\begin{aligned} dV &= N_S(\mathrm{d}S + \delta S\,\mathrm{d}t) + N_\beta\,\mathrm{d}\beta \\ &= N_S(\mathrm{d}S + \delta S\,\mathrm{d}t) + r(N_\beta\beta)r\,\mathrm{d}t \\ &= N_S(\mathrm{d}S + \delta S\,\mathrm{d}t) + r(V - N_S S)\,\mathrm{d}t \\ &= (rV - (r-\delta)N_S S)\,\mathrm{d}t + N_S\,\mathrm{d}S, \end{aligned} \tag{3}$$

where in the second line we used the fact that $\mathrm{d}\beta = r\beta\,\mathrm{d}t$, and in the third line we applied the self-financing condition (2) re-written as $N_\beta\beta = V - N_S S$.

As in the previous chapter, equation (3) captures the dynamics of the replicating portfolio needed to hedge the short position. For the hedge to be successful, the dynamics of the long position must match the dynamics of the short position that are given by:

$$\mathrm{d}V = \frac{\partial V}{\partial S}\,\mathrm{d}S + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\,\mathrm{d}t + \frac{\partial V}{\partial t}\,\mathrm{d}t. \tag{4}$$

Equating (3) and (4) shows that:

$$\left(\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t}\right)\mathrm{d}t + \frac{\partial V}{\partial S}\,\mathrm{d}S = (rV - (r-\delta)N_S S)\,\mathrm{d}t + N_S\,\mathrm{d}S. \tag{5}$$

Equation (5) reveals that, even in the presence of a dividend yield, the number of shares required to hedge the option must equal the partial derivative with respect to the stock price. Therefore, choosing $N_S = \frac{\partial V}{\partial S}$ implies that:

$$\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r-\delta)S\frac{\partial V}{\partial S} + \frac{\partial V}{\partial t} - rV = 0 \tag{6}$$

with boundary condition $V_T = F(S_T)$.

Equation (6) is the Black-Scholes partial differential equation (PDE) that must satisfy all derivatives written on an asset that pays a dividend yield.[1]

Example 1. Consider a forward contract written on an asset that pays a dividend yield q expiring at time T with forward price K. The value V of the forward contract at time $0 \leq t \leq T$ is:

$$V = Se^{-\delta(T-t)} - Ke^{-r(T-t)}$$

Let us check that the value of the contract satisfies the Black-Scholes PDE equation (6).

[1] The Black-Scholes PDE given by (6) is only valid for derivative instruments (or assets) that do not pay dividends themselves, even though they are written on a dividend paying asset. The stock, for example, does not satisfy equation (6) since it pays a dividend. A generalized PDE that allows for the derivative to also pay a dividend yield can be found in the appendix.

The relevant derivatives are

$$\frac{\partial V}{\partial S} = e^{-\delta(T-t)},\ \frac{\partial^2 V}{\partial S^2} = 0,$$
$$\frac{\partial V}{\partial t} = \delta S e^{-\delta(T-t)} - rKe^{-r(T-t)}.$$

The left-hand side of equation (6) then becomes

$$(r-\delta)Se^{-\delta(T-t)} + \delta S e^{-\delta(T-t)} - rKe^{-r(T-t)} - r(Se^{-\delta(T-t)} - Ke^{-r(T-t)}) = 0,$$

which clearly satisfy the claim. □

The Risk-Neutral Process for the Underlying Asset

As in the Black-Scholes model, the replication argument is indifferent of the dynamics of the stock. This implies that the same logic should work in a hypothetical world where everyone is risk-neutral. In such a world, the *expected total return* of all assets is the risk-free rate. Hence, equation (1) implies that:

$$\frac{S_{t+\Delta t} + \delta S_t \Delta t - S_t}{S_t} = \frac{\Delta S_t}{S_t} + \delta \Delta t = r\Delta t + \sigma \Delta B_t^*.$$

As $\Delta t \to 0$ we obtain the continuous-time analog of risk-neutral process followed by the stock:

$$\mathrm{d}S = (r-\delta)S\,\mathrm{d}t + \sigma S\,\mathrm{d}B^*.$$

We conclude that S follows a GBM under the risk-neutral measure with drift $r-\delta$ and volatility σ.

Example 2. We can use the risk-neutral approach to compute the value V of a long forward contract with maturity T and forward price K. The payoff of the long forward at maturity is given by $S_T - K$.

The value of the contract is then the expected payoff discounted at the risk-free rate, i.e.,

$$\begin{aligned} V &= e^{-rT}\,\mathsf{E}^*(S_T - K) \\ &= e^{-rT}(Se^{(r-\delta)T} - K) \\ &= Se^{-\delta T} - Ke^{-rT}. \end{aligned}$$

The value of the forward, in general, will change over time. The forward price F is determined such that the value of the contract is zero when the contract is signed. Thus,

$$V = Se^{-\delta T} - Fe^{-rT} = 0 \Rightarrow F = Se^{(r-\delta)T}.$$

□

Remember that under the risk-neutral measure, the value of any asset is computed as its expected payoff discounted at the risk-free rate. Therefore, the price of a European call option with maturity T and strike price K written on the asset that pays a continuous dividend yield q is given by:

$$\begin{aligned} C &= e^{-rT}\,\mathsf{E}^*\left((S_T - K)\mathbf{1}_{\{S_T > K\}}\right) \\ &= e^{-rT}\,\mathsf{E}^*\left(S_T\mathbf{1}_{\{S_T > K\}}\right) - e^{-rT}\,\mathsf{E}\left(K\mathbf{1}_{\{S_T > K\}}\right) \\ &= Se^{-\delta T}\,\Phi(d_1) - Ke^{-rT}\,\Phi(d_2), \end{aligned}$$

where

$$\begin{aligned} d_1 &= \frac{\ln(S/K) + (r - \delta + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}, \\ d_2 &= d_1 - \sigma\sqrt{T}. \end{aligned}$$

Note that, even though the risk-free rate is positive, the time-value of deep ITM call options is now negative due to the positive dividend yield. Indeed, the lower bound asymptote has a slope coefficient less than one, making the option price to cross the option's intrinsic value.

Consider now a European put option with the same characteristics as the previous call.

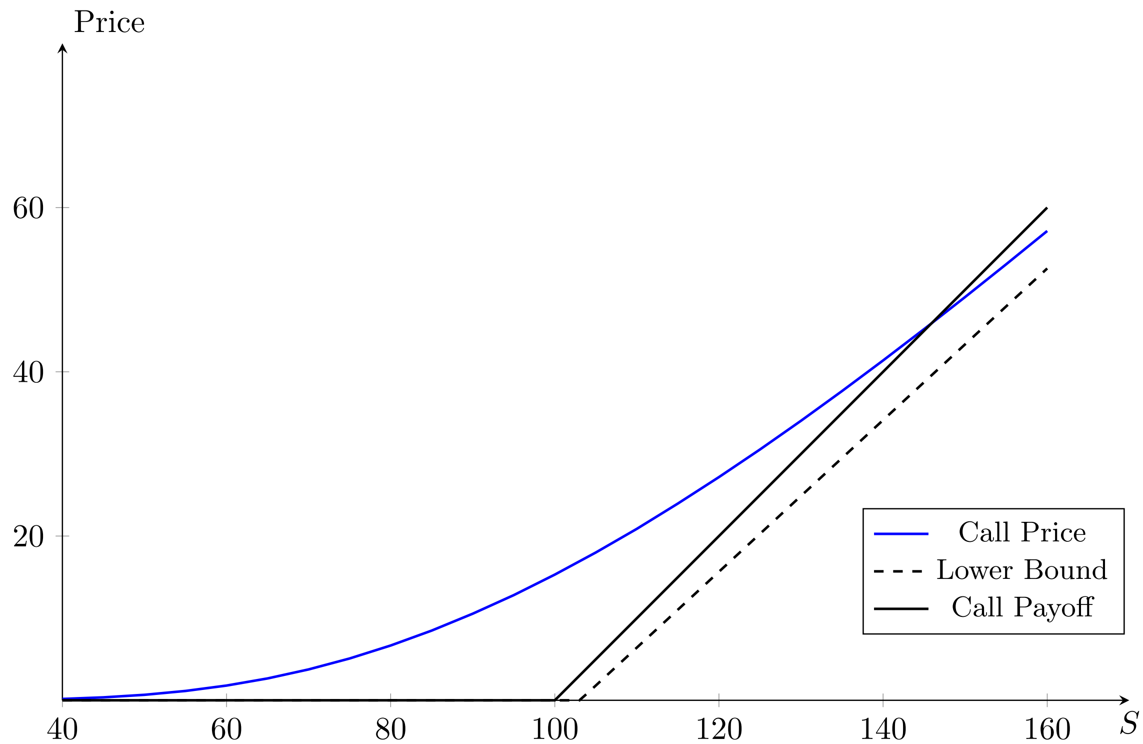


Figure 1: The figure displays the Black-Scholes call premium $C(S)$ where $r = 0.05$, $\delta = 0.08$, $\sigma = 0.45$, $T = 1$ and $K = 100$. It also shows the call option payoff given by $\max(S-K, 0)$ and the lower bound for a European call given by $\max(Se^{-\delta T} - Ke^{-rT}, 0)$.

According to put-call parity, it must be the case that

$$C - P = Se^{-\delta T} - Ke^{-rT}.$$

Hence,

$$\begin{aligned}
P &= C - (Se^{-\delta T} - Ke^{-rT}) \\
&= Se^{-\delta T}\,\Phi(d_1) - Ke^{-rT}\,\Phi(d_2) - (Se^{-\delta T} - Ke^{-rT}) \\
&= Ke^{-rT}(1 - \Phi(d_2)) - Se^{-\delta T}(1 - \Phi(d_1)) \\
&= Ke^{-rT}\,\Phi(-d_2) - Se^{-\delta T}\,\Phi(-d_1).
\end{aligned}$$

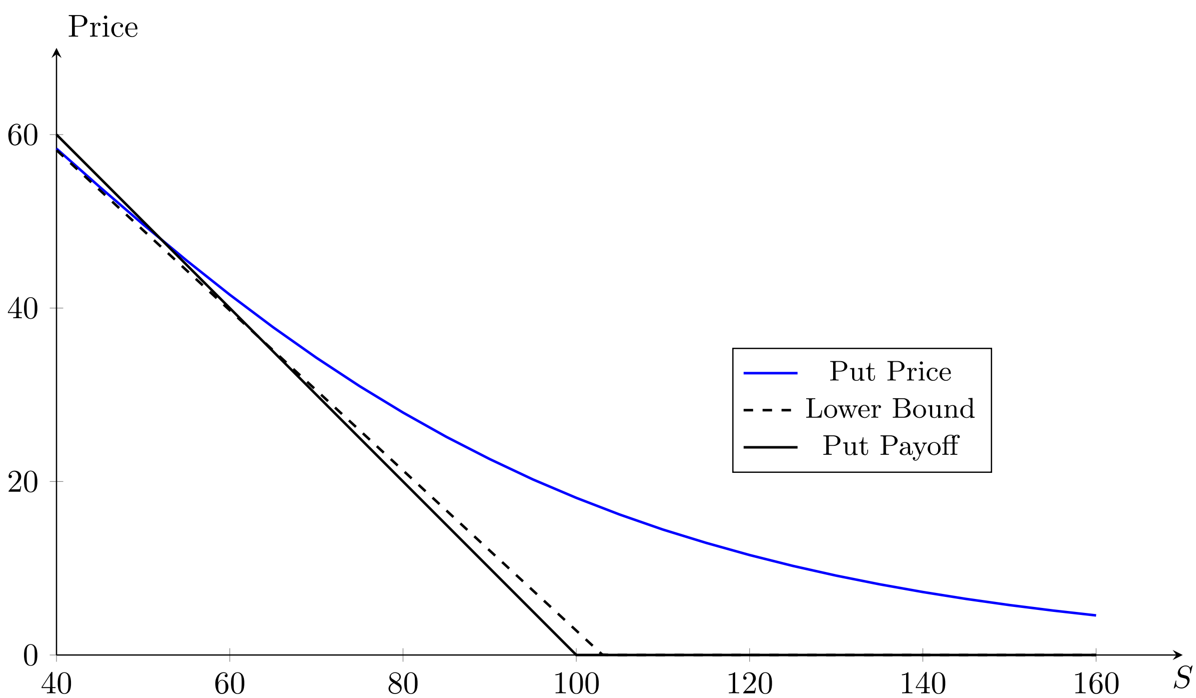


Figure 2: The figure displays the Black-Scholes put premium $P(S)$ where $r = 0.05$, $\delta = 0.08$, $\sigma = 0.45$, $T = 1$ and $K = 100$. It also shows the put option payoff given by $\max(K - S, 0)$ and the lower bound for a European put given by $\max(Ke^{-rT} - Se^{-\delta T}, 0)$.

Example 3. A stock that pays a continuous dividend yield of 8% currently trades for \$100. The instantaneous volatility of returns is 30% per year and the risk-free rate is 5% per year, continuously compounded and constant for all maturities. Consider ATM call and put options written on the stock with maturity 10 months. Then,

$$d_1 = \frac{\ln(100/100) + (0.05 - 0.08 + 0.5(0.30)^2)(10/12)}{0.30\sqrt{10/12}} = 0.0456,$$
$$d_2 = 0.0456 - 0.30\sqrt{10/12} = -0.2282.$$

Therefore, $\Phi(d_1) = 0.5182$ and $\Phi(d_2) = 0.4097$, which implies that

$$C = 100e^{-0.08(10/12)}(0.5182) - 100e^{-0.05(10/12)}(0.4097) = \$9.18,$$
$$P = 100e^{-0.05(10/12)}(1 - 0.4097) - 100e^{-0.08(10/12)}(1 - 0.5182) = \$11.54.$$

□

For an asset that a pays a continuous dividend yield δ, we have that for a European call option:

$$\frac{\partial C}{\partial S} = e^{-\delta T}\Phi(d_1).$$

We can see that if $\delta > 0$, the number of shares required to hedge the call is lower than in the case of a non-dividend paying asset. The shares that you buy to hedge the call grow over time at the rate δ, which means that you need to buy less.

Similarly, for a European put option we have that

$$\frac{\partial P}{\partial S} = -e^{-\delta T}\Phi(-d_1).$$

Example 4. In the previous example, we found that $\Phi(d_1) = 0.5182$ and $\Phi(d_2) = 0.4097$. Hence,

$$\frac{\partial C}{\partial S} = e^{-0.08(10/12)}(0.5182) = 0.4848,$$
$$\frac{\partial P}{\partial S} = -e^{-0.08(10/12)}(1 - 0.5182) = -0.4507.$$

This means that an OTC dealer who sells a call option needs to buy 0.4848 units of the asset while borrowing

$$100e^{-0.05(10/12)}(0.4097) = \$39.30$$

at the risk-free rate. To hedge a put option, the dealer needs to short-sell 0.4507 units of the asset and invest

$$100e^{-0.05(10/12)}(1-0.4097) = \$56.62$$

in the money-market account. □

Black-Scholes Model for a Stock that Pays a Dividend Yield

Consider a stock S that pays a continuous dividend yield δ and that follows a GBM under the risk-neutral measure:

$$\mathrm{d}S = (r-\delta)S\,\mathrm{d}t + \sigma S\,\mathrm{d}B^*.$$

The price of European call and put options with strike price K and time-to-maturity T are given by:

$$\begin{aligned} C &= Se^{-\delta T}\,\Phi(d_1) - Ke^{-rT}\,\Phi(d_2), \\ P &= Ke^{-rT}\,\Phi(-d_2) - Se^{-\delta T}\,\Phi(-d_1), \end{aligned}$$

where

$$\begin{aligned} d_1 &= \frac{\ln(S/K) + (r-\delta+\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}, \\ d_2 &= d_1 - \sigma\sqrt{T}. \end{aligned}$$

Furthermore, the delta of the call is given by $e^{-\delta T}\,\Phi(d_1)$ whereas the delta of the put is computed as $-e^{-\delta T}\,\Phi(-d_1)$.

Options on Indices

Options on Stock Indices

Most stock indices such as the S&P 500 (SPX) do not reinvest their dividends. Hence, to replicate an option written on the index we can use a portfolio of stocks that mimics the value of the index and that will pay a dividend yield over time. We will assume that the replicating portfolio exactly matches the composition of the index at any point in time so that S_t represents both the value of the index and of the tracking portfolio.

SPX Options

SPX options are one of the most liquid option contracts in the world. They have the following characteristics:

- European style exercise
- Cash settled
- Each contract is written on 100 times the value of the index

There are also mini-SPX index options written over XSP which is an index 10 times smaller than SPX. More information can be found at https://cdn.cboe.com/resources/spx/spx-fact-sheet.pdf.

Example 5. The SPX index is currently at 4,251, has a dividend yield of 1.33% per year and an instantaneous volatility of 17% per year. The risk-free rate is 3% per year, continuously compounded and constant for all maturities. Say we want to compute the price of an SPX call option contract with maturity 3 months and strike 4,300. Then,

$$d_1 = \frac{\ln(4251/4300) + (0.03 - 0.0133 + 0.5(0.17)^2)(3/12)}{0.17\sqrt{3/12}} = -0.0432,$$
$$d_2 = -0.0432 - 0.17\sqrt{10/12} = -0.1282.$$

Hence, $\Phi(d_1) = 0.4828$ and $\Phi(d_2) = 0.4490$, which implies that:

$$C = 4,251e^{-0.0133(3/12)}(0.4828) - 4,300e^{-0.03(3/12)}(0.4490) = \$129.193.$$

Therefore, a standard SPX call option contract should cost $12,919.30, whereas a mini-SPX call option contract should trade for $1,291.93. □

Appendix

General PDE for Dividend-Paying Derivatives

Consider a derivative H expiring at time T and written on an asset that pays a dividend yield δ_S. Let us assume that the derivative itself also pays a dividend yield δ_H. The risk-neutral process for the stock is given by:

$$\mathrm{d}S = (r - \delta_S)S\,\mathrm{d}t + \sigma_S S\,\mathrm{d}B^*,$$

whereas the risk-neutral process of the derivative is given by

$$\mathrm{d}H = (r - \delta_H)H\,\mathrm{d}t + \sigma_H H\,\mathrm{d}B^*. \tag{7}$$

Moreover, according to Ito's lemma, the risk-neutral process of the derivative must also satisfy:

$$\begin{aligned} dH &= \frac{\partial H}{\partial S}\,\mathrm{d}S + \frac{1}{2}\frac{\partial^2 H}{\partial S^2}(\mathrm{d}S)^2 + \frac{\partial H}{\partial t}\,\mathrm{d}t \\ &= \left(\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 H}{\partial S^2} + (r - \delta_S)S\frac{\partial H}{\partial S} + \frac{\partial H}{\partial t}\right)\mathrm{d}t + \sigma S\frac{\partial H}{\partial S}\,\mathrm{d}B^*. \end{aligned} \tag{8}$$

Since the drift in (7) and (8) is the same, we have that

$$\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 H}{\partial S^2} + (r - \delta_S)S\frac{\partial H}{\partial S} + \frac{\partial H}{\partial t} - (r - \delta_H)H = 0. \tag{9}$$

In particular, the stock itself satisfies (9) if we take $H(S) = S$ and $\delta_H = \delta_S$.

Practice Problems

Solutions to all problems can be found at lorenzonaranjo.com/fin5241-fall25b.

Problem 1. Calculate the value of a three-month at-the-money European call option on a stock index when the index is at 250, the risk-free interest rate is 10% per annum, the volatility of the index is 18% per annum, and the dividend yield on the index is 3% per annum.

Problem 2. The S&P 100 index currently stands at 696 and has a volatility of 30% per annum. The risk-free rate of interest is 7% per annum and the index provides a dividend yield of 4% per annum. Calculate the value of a three-month European put with strike price 700.