Prof. Lorenzo Naranjo

Modeling Stock Prices in Continuous-Time

Stochastic Processes

A stochastic process describes the evolution of a random variable over time. In finance, we use stochastic processes to model the evolution of stock prices, interest rates, volatility, foreign exchange rates, and commodity prices. We distinguish between:

- *Discrete-time processes*: The values of the process $\{S_n\}$ are allowed to change only at discrete time intervals, i.e., $n \in \{0, 1, 2, \dots, N\}$ or $n \in \mathbb{N}$.
- *Continuous-time processes*: The stochastic process $\{S_t\}$ is defined for all $t \in [0, T]$.

We will now consider several stochastic processes commonly used to model the future evolution of the price of an asset such as a stock. We start by understanding discrete-time processes and then extend the analysis to include continuous-time processes. The analysis is informal, as the theory of stochastic process in continuous time requires advanced mathematical concepts, which is beyond the scope of these notes.

It is essential to realize that a stochastic process for a stock price is trying to model all possible *histories* between now and a specific time in the future. A *sample path* is one of the many possible histories generated using the stochastic process.

Random Walks

We will now study one of the simplest yet most intriguing stochastic processes defined in discrete time. A one-dimensional *random walk* $\{X_n\}$ is a stochastic process defined as

$$
\begin{aligned}
X_0 &= x_0, \\
X_{n+1} &= X_n + e_{n+1},
\end{aligned}
$$

where $\{e_n\}$ are independent and identically distributed (i.i.d.) random variables such that $\mathrm{E}(e_n) = 0$ for all $n \geq 1$. Note that e_n need not be normally distributed. For example, for each n, the variable e_n could take the values 1 and -1 with equal probability. A random walk only requires that the shocks e_n are independent.

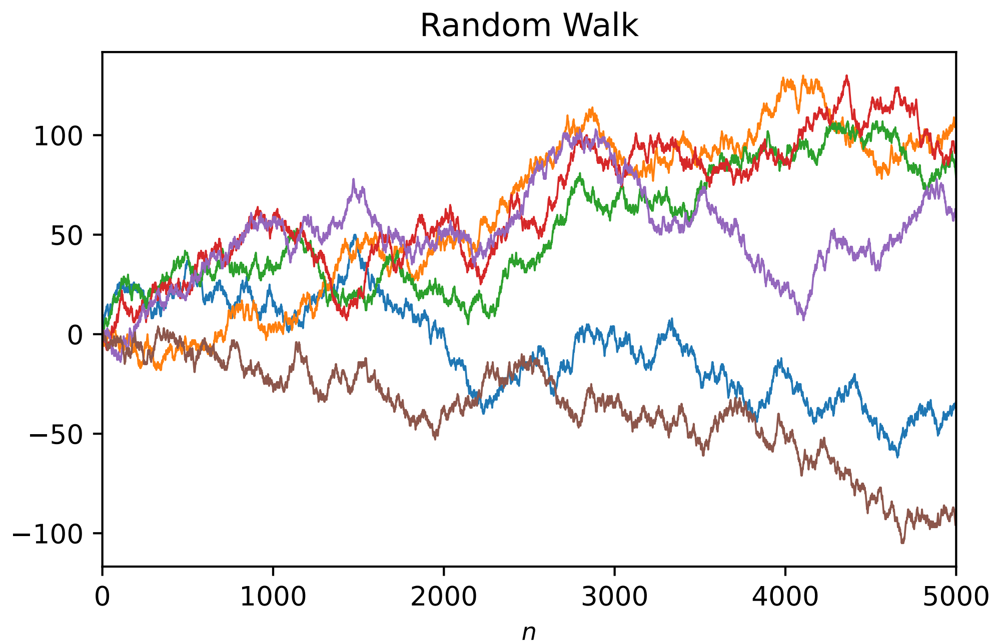


Figure 1: The figure plots simulated paths for the random walk defined as $X_0 = 0, X_{n+1} = X_n + e_{n+1}$, where $\{e_n\}$ is an i.i.d sequence taking the values 1 and -1 with equal probability, and $1 \leq n \leq 5000$.

An essential property of a random walk is that its sample paths diverge as n grows. Indeed, we have

$$\begin{aligned} X_n &= X_{n-1} + e_n \\ &= X_{n-2} + e_{n-1} + e_n \\ &\;\;\vdots \\ &= X_0 + e_1 + \cdots + e_{n-1} + e_n \\ &= X_0 + \sum_{i=1}^{n} e_i. \end{aligned}$$

Denoting $\mathrm{V}(e_n) = \sigma^2$, and since we have that $\{e_n\}$ are independent, we have

$$\mathrm{V}(X_n) = n\sigma^2.$$

Therefore, the variance of X_n increases linearly with n as $n \to \infty$.

Intuitively, this is saying that if we simulate many different sample paths for $n = 0, \dots, N$ where N is very large, we should expect to see some values of X_N to be very high and positive whereas others will be significantly negative.

Brownian Motion

A very useful random walk can be defined as follows:

$$B_{t+\Delta t} = B_t + \sqrt{\Delta t} e_{t+\Delta t},$$

where $B_0 = 0$ and $\{e_t\}$ are i.i.d. such that $e_t \sim N(0,1)$. Note that here time increases each step by Δt. Letting $\Delta t \to 0$, the resulting process $\{B_t\}$ for $t \in [0,T]$ is called a Brownian motion or Wiener process.

The Brownian motion has the following properties:

- The sample paths are continuous.
- For $s < t$, the increment $B_t - B_s \sim N(0, t-s)$, i.e. is normally distributed with mean 0 and variance $t - s$.
- Increments are independent of each other.
- In particular, note that $B_t \sim N(0,t)$ for $0 < t \leq T$.

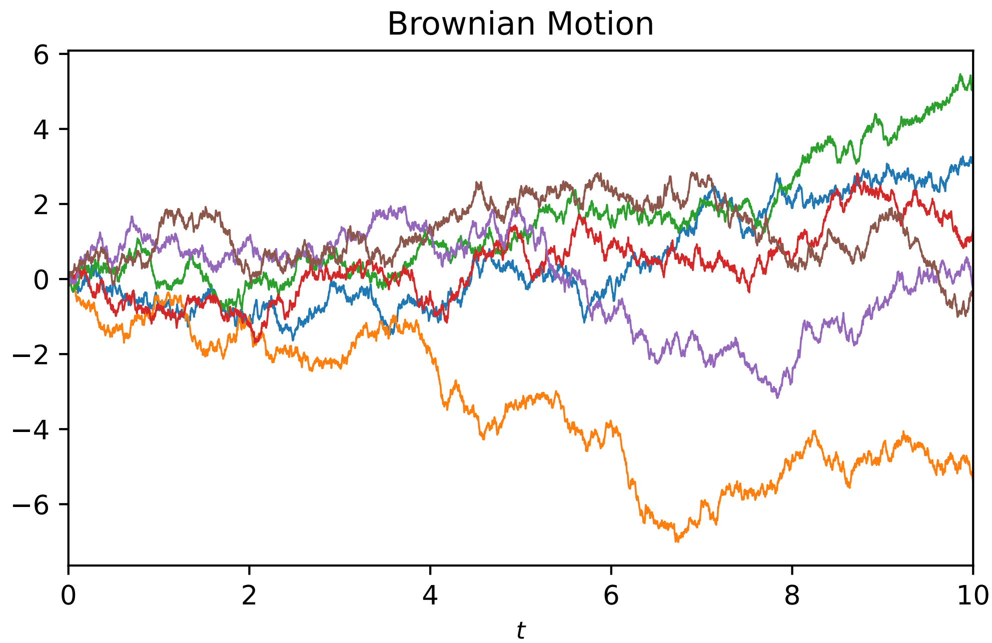


Figure 2: The figure plots simulated paths for $\{B_t\}$ where $0 \le t \le 10$.

Geometric Brownian Motion

Now we turn our attention to modeling stock prices $\{S_t\}$. We need to be careful, though, as stock prices cannot be negative. We also would like to allow the model to display a certain drift μ and volatility σ.

To achieve this, we model the percentage change of a stock price between t and $t + \Delta t$ as

$$\frac{\Delta S_t}{S_t} = \mu \Delta t + \sigma \Delta B_t.$$

Note that the percentage change in price over an interval Δt is normally distributed with mean $\mu \Delta t$ and variance $\sigma^2 \Delta t$. Letting $\Delta t \to 0$, the resulting process $\{S_t\}$ for $t \in [0, T]$ is called a geometric Brownian motion (GBM).

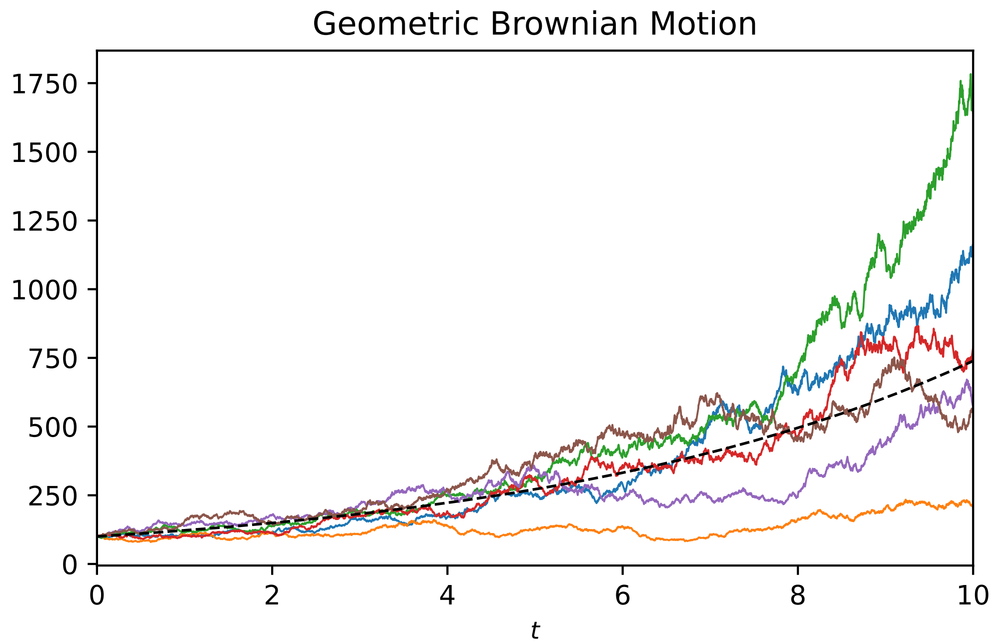


Figure 3: The figure plots simulated paths for a geometric Brownian motion $\{S_t\}$ where $0 \leq t \leq 10$, $S_0 = 100$, $\mu = 0.20$, and $\sigma = 0.20$. The dashed line denotes $\mathrm{E}\left(S_t\right) = S_0 e^{\mu t}$.

Stochastic Calculus

Once we have defined how S_t behaves over time, we now turn our attention to model how a *function* of S_t behaves over time. The reason why we are interested in this is because we want to find a way to price derivatives as a function of the relevant state variables. We will see later that when the stock price is driven by a single source of uncertainty, then the value of a call or put option depends only on the stock price itself and time-to-maturity, i.e. the price of the derivative when the stock price is S and the time-to-maturity is T will be of the form $F(S, T)$.

We will start studying how $X_t = F(S_t)$ behaves over time and we will add later the time dimension to the problem. In what follows we assume that $F(\cdot)$ is a smooth function such that its first and second derivatives exist.

Ito's Lemma

Remember that the Brownian motion increment is defined

$$\Delta B_t = B_{t+\Delta t} - B_t = \sqrt{\Delta t} e_{t+\Delta t}.$$

Consider a GBM process $\{S_t\}$ and a smooth function $F(\cdot)$. A second order Taylor approximation around S_t implies

$$F(S_t + \Delta S_t) \approx F(S_t) + F'(S_t)(\Delta S_t) + \frac{1}{2} F''(S_t)(\Delta S_t)^2.$$

Using the results derived in the appendix, we have that

$$\begin{aligned}
(\Delta S_t)^2 &= (\mu S_t \Delta t + \sigma S_t \Delta B_t)^2 \\
&= (\mu S_t)^2 \underbrace{(\Delta t)^2}_{\approx 0} + 2\mu\sigma (S_t)^2 \underbrace{(\Delta t)(\Delta B_t)}_{\approx 0} + (\sigma S_t)^2 \underbrace{(\Delta B_t)^2}_{\approx \Delta t} \\
&\approx \sigma^2 S_t^2 \Delta t.
\end{aligned}$$

We can finally conclude that

$$\Delta F(S_t) \approx \left(\mu S_t F'(S_t) + \frac{1}{2} \sigma^2 S_t^2 F''(S_t) \right) \Delta t + \sigma S_t F'(S_t) \Delta B_t.$$

The continuous-time analog of the previous analysis is as follows.

Ito's Lemma for GBM

Consider a GBM process $\{S_t\}$ given by

$$\mathrm{d}S = \mu S \,\mathrm{d}t + \sigma S \,\mathrm{d}B, \tag{1}$$

and a twice-differentiable function $F(S)$. Then we have

$$\mathrm{d}F = \left(\mu S F'(S) + \frac{1}{2} \sigma^2 S^2 F''(S) \right) \mathrm{d}t + \sigma S F'(S) \,\mathrm{d}B.$$

It is usually more convenient to use the *box* calculus when working with stochastic processes defined through Brownian motions.

Box Calculus

Consider the GBM process $\{S_t\}$ defined in (1). The *box* calculus rules for Ito processes are:

$$
\begin{aligned}
(\mathrm{d}t)^2 &= 0, \\
(\mathrm{d}t)(\mathrm{d}B) = (\mathrm{d}B)(\mathrm{d}t) &= 0, \\
(\mathrm{d}B)^2 &= \mathrm{d}t.
\end{aligned}
$$

Furthermore, denote $F_S = F'(S)$ and $F_{SS} = F''(S)$. Ito's Lemma can then be restated as

$$
\mathrm{d}F = F_S \,\mathrm{d}S + \frac{1}{2} F_{SS} (\mathrm{d}S)^2,
$$

where

$$
(\mathrm{d}S)^2 = (\mu S \,\mathrm{d}t + \sigma S \,\mathrm{d}B)^2 = \sigma^2 S^2 \,\mathrm{d}t.
$$

Solving for GBM

Define $X = \ln(S)$, which implies $S = e^X$. We have that $F_S = 1/S$ and $F_{SS} = -1/S^2$, which implies

$$
\begin{aligned}
dX &= F_S \,\mathrm{d}S + \frac{1}{2} F_{SS} (\mathrm{d}S)^2 \\
&= \frac{1}{S} (\mu S \,\mathrm{d}t + \sigma S \,\mathrm{d}B) + \frac{1}{2} \left(-\frac{1}{S^2} \right) \sigma^2 S^2 \,\mathrm{d}t \\
&= (\mu \,\mathrm{d}t + \sigma \,\mathrm{d}B) - \frac{1}{2} \sigma^2 \,\mathrm{d}t \\
&= \left(\mu - \frac{1}{2} \sigma^2 \right) \mathrm{d}t + \sigma \,\mathrm{d}B.
\end{aligned}
$$

We can then solve for X_T:

$$
\begin{aligned}
X_T - X_0 = \int_0^T dX &= \int_0^T \left(\mu - \frac{1}{2} \sigma^2 \right) \mathrm{d}t + \int_0^T \sigma \,\mathrm{d}B \\
&= \left(\mu - \frac{1}{2} \sigma^2 \right) T + \sigma B_T,
\end{aligned}
$$

and conclude that

$$S_T = S_0 \exp\left(\left(\mu - \frac{1}{2}\sigma^2\right)T + \sigma B_T\right). \tag{2}$$

Properties of Stock Prices Following a GBM

Equation (2) can be rewritten as:

$$\ln(S_T) = \ln(S_0) + \left(\mu - \frac{1}{2}\sigma^2\right)T + \sigma B_T.$$

We can conclude that $\ln(S_T) \sim N(m, s^2)$, where

$$\begin{aligned} m &= \ln(S_0) + \left(\mu - \frac{1}{2}\sigma^2\right)T, \\ s &= \sigma\sqrt{T}. \end{aligned}$$

In other words, S_T is lognormally distributed with mean m and variance s^2.

Example 1. Consider a stock whose price at time t is given by S_t and that follows a GBM. The expected return is 12% per year and the volatility is 25% per year. The current spot price is \$25. If we denote $X_T = \ln(S_T)$ and take $T = 0.5$, we have that:

$$\begin{aligned} \mathrm{E}(X_T) &= \ln(25) + \left(0.12 - 0.5(0.25)^2\right)(0.5) = 3.2633, \\ \mathrm{SD}(X_T) &= 0.25\sqrt{0.5} = 0.1768. \end{aligned}$$

Hence, the 95% confidence interval for S_T is given by:

$$[e^{3.2633-1.96(0.1768)}, e^{3.2633+1.96(0.1768)}] = [18.48, 36.96].$$

Therefore, there is a 95% probability that the stock price in 6 months will lie between \$18.48 and \$36.96. □

Moments of the Stock Price

The fact that the stock price at time T is log-normally distributed allows us to compute the mean and standard deviation of S_T.

> Moments of the Stock Price
>
> The expectation and standard deviation of S_T are given by:
>
> $$\mathsf{E}(S_T) = S_0 e^{\mu T},$$
> $$\mathsf{SD}(S_T) = E(S_T)\sqrt{e^{\sigma^2 T} - 1}.$$

Proof

Since $\ln(S_T) \sim \mathcal{N}(m, s^2)$, we can compute its moments using the results derived earlier so that

$$\mathsf{E}(S_T) = e^{m+\frac{1}{2}s^2} = e^{\ln(S_0)+\left(\mu-\frac{1}{2}\sigma^2\right)T+\frac{1}{2}\sigma^2 T} = e^{\ln(S_0)}e^{\mu T} = S_0 e^{\mu T}.$$

In this model, the expected stock price at any point in the future is just the current stock price growing at the rate μ for T years. $\square$

Therefore, the expected stock price grows at a rate μ. The variance of S_T, however, is large and increases exponentially with time.

Example 2. Consider a stock whose price at time t is given by S_t and that follows a GBM. The expected return is 12% per year and the volatility is 25% per year. The current spot price is $25. The expected price and standard deviation 6 months from now are:

$$\mathsf{E}(S_T) = 25e^{0.12(0.5)} = \$26.55,$$
$$\mathsf{SD}(S_T) = 26.55\sqrt{e^{0.25^2(0.5)} - 1} = \$4.73.$$

$\square$

A Generalized Form of Ito's Lemma

Most derivatives not only depend on the underlying asset but also depend on time since they have fixed expiration dates. The analysis we did before for Ito's Lemma generalizes easily to handle this case. Consider a non-dividend paying stock that follows a GBM:

$$\mathrm{d}S = \mu S\,\mathrm{d}t + \sigma S\,\mathrm{d}B,$$

and a smooth function $F(S,t)$. Ito's Lemma in this case applies in the following form:

$$\mathrm{d}F = F_S\,\mathrm{d}S + \frac{1}{2}F_{SS}(\mathrm{d}S)^2 + F_t\,\mathrm{d}t,$$

where $(\mathrm{d}S)^2 = \sigma^2 S^2\,\mathrm{d}t$.

Appendix

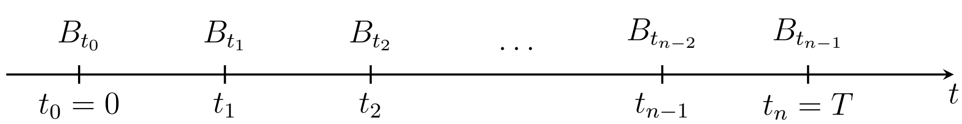
Some Intuition on Brownian Motion

Remember that we defined the Brownian motion or Wiener process as a random walk driven by normally distributed shocks:

$$B_{t+\Delta t} = B_t + \sqrt{\Delta t}e_{t+\Delta t},$$

where $\{e_t\}$ is an i.i.d. sequence of random variables distributed $\mathcal{N}(0,1)$.

Let's start by splitting the interval $[0,T]$ into n intervals of length $\Delta t = t_{i+1} - t_i$.



Note that $t_i = i\Delta t$ and $T = t_n = n\Delta t$. The Brownian motion increments are then defined as $\Delta B_{t_i} = B_{t_{i+1}} - B_{t_i}$.

The first question one might have is why using normally distributed increments. There are two answers for that. First, a sum of normally distributed random variables is also normal and in this case we have:

$$B_T - B_0 = \sum_{i=0}^{n-1} \Delta B_{t_i} = \sum_{i=0}^{n-1} \sqrt{\Delta t} e_{t+\Delta t}.$$

The variance of $\sum_{i=0}^{n-1} \sqrt{\Delta t} e_{t+\Delta t}$ is given by $\sum_{i=0}^{n-1} \Delta t = n\Delta t = T$, which implies that $B_T \sim \mathcal{N}(0, T)$. So by using normally distributed increments we guarantee that the resulting process for Brownian motion is also normal.

Second, imagine that we use a different distribution for the i.i.d. increments while still requiring $\mathrm{E}(e_t) = 0$ and $\mathrm{V}(e_t) = 1$. For example, e_t could take the values 1 and -1 with equal probability. Nevertheless, the central limit theorem guarantees that:

$$\sqrt{n}\left(\frac{1}{n}\sum_{i=0}^{n-1} \sqrt{\Delta t} e_{t+\Delta t}\right) \xrightarrow{d} \mathcal{N}(0, \Delta t).$$

In other words, even if we use a different distribution for the increments, as $n \to \infty$ we have that $B_T \sim \mathcal{N}(0, T)$. Therefore, there is no loss in generality in assuming normally distributed increments for the Brownian motion.

A second question that one might have, and one of the most puzzling facts in stochastic calculus in my opinion, is the fact that when we apply Ito's lemma we use the fact that $(dB_t)^2 = \mathrm{d}t$. Clearly, $(\Delta B_t)^2 = \Delta t e_t^2 \neq \Delta t$ where $e_t \sim \mathcal{N}(0,1)$. Indeed, if ΔB_t is random, then $(\Delta B_t)^2$ must also be random. However, we will see in a moment that it is fine to say that $(\Delta B_t)^2 \approx \Delta t$ as $\Delta t \to 0$.

Let's start by computing the mean and variance of $(\Delta B_t)^2$:

$$
\begin{aligned}
\mathsf{E}\left[(\Delta B_t)^2\right] &= \Delta t \\
\mathsf{V}\left[(\Delta B_t)^2\right] &= \mathsf{E}\left[(\Delta B_t)^4\right] - \left(\mathsf{E}\left[(\Delta B_t)^2\right]\right)^2 \\
&= 3(\Delta t)^2 - (\Delta t)^2 \\
&= 2(\Delta t)^2.
\end{aligned}
$$

In computing the variance of $(\Delta B_t)^2$ we used the fact that if $X \sim \mathcal{N}(0, \sigma^2)$, then $\mathsf{E}(X^4) = 3\sigma^4$. Since $\Delta B_t \sim \mathcal{N}(0, \Delta t)$, we have that $\mathsf{E}\left[(\Delta B_t)^4\right] = 3(\Delta t)^2$.

Consider now the following sum:

$$
S_n = \sum_{i=0}^{n-1} (\Delta B_{t_i})^2.
$$

Clearly, S_n is a sum of n independent random variables so its variance is the sum of the variance of each ΔB_t :

$$
\begin{aligned}
\mathsf{E}(S_n) &= n\Delta t = T \\
\mathsf{V}(S_n) &= n(2(\Delta t)^2) = \frac{2T^2}{n}.
\end{aligned}
$$

Since $\lim_{n\to\infty} \mathsf{V}(S_n) = 0$, we have that $S_n \to T$ as $n \to \infty$ in probability. Intuitively, the previous result is really the weak-law of large numbers since we can re-write it as $\frac{S_n}{n} \to \Delta t$ as $n \to \infty$ in probability. However, when you apply the weak-law of large numbers to an arbitrary sequence of i.i.d. random variables, you cannot say that you can approximate each random variable by its mean just because its average converges to their mean. In our case, since the variance of $(\Delta B_t)^2$ is so small compared to its mean, we can safely say that $(\Delta B_t)^2$ behaves as if $(\Delta B_t)^2 = \Delta t$ as $n \to \infty$. In other words, we have that $(\Delta B_t)^2 \approx \Delta t$ for small Δt.

We can apply the same analysis to study the behavior of $(\Delta t)(\Delta B_t)$ as $\Delta t \to 0$. Since:

$$\begin{aligned}
\mathsf{E}\left[(\Delta t)(\Delta B_t)\right] &= 0 \\
\mathsf{V}\left[(\Delta t)(\Delta B_t)\right] &= \mathsf{E}\left[((\Delta t)(\Delta B_t))^2\right] - (\mathsf{E}[(\Delta t)(\Delta B_t)])^2 \\
&= (\Delta t)^2\,\mathsf{E}[(\Delta B_t)^2] - ((\Delta t)\,\mathsf{E}[\Delta B_t])^2 \\
&= (\Delta t)^3.
\end{aligned}$$

Consider now the following sum:

$$C_n = \sum_{i=0}^{n-1} (\Delta t)(\Delta B_{t_i}).$$

The mean and variance of C_n are given by:

$$\begin{aligned}
\mathsf{E}(C_n) &= 0 \\
\mathsf{V}(C_n) &= \frac{T^3}{n^2}.
\end{aligned}$$

Since $\lim_{n\to\infty} \mathsf{V}(C_n) = 0$, we have that $C_n \to 0$ as $n \to \infty$ in probability, implying that $(\Delta t)(\Delta B_t) \approx 0$ for small Δt.

Computing Partial Expectations

Since $\ln(S_T) \sim \mathcal{N}(m, s^2)$, we can use the result introduced earlier about partial expectations to show the following property.

Partial Expectations of the Stock Price

Consider a non-dividend paying stock that follows a GBM as defined in 1. Then we have that:

$$\begin{aligned} \mathrm{E}\left(S_T \mathbf{1}_{\{S_T>K\}}\right) &= S_0 e^{\mu T}\,\Phi\left(\frac{\ln(S_0/K)+(\mu+\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right), \\ \mathrm{E}\left(K \mathbf{1}_{\{S_T>K\}}\right) &= K\,\Phi\left(\frac{\ln(S_0/K)+(\mu-\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right). \end{aligned}$$

Proof

$$\begin{aligned} \mathrm{E}\left(S_T \mathbf{1}_{\{S_T>K\}}\right) &= e^{m+\frac{1}{2}s^2}\,\Phi\left(\frac{m+s^2-\ln(K)}{s}\right) \\ &= S_0 e^{\mu T}\,\Phi\left(\frac{\ln(S_0/K)+(\mu+\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right), \\ \mathrm{E}\left(K \mathbf{1}_{\{S_T>K\}}\right) &= K\,\Phi\left(\frac{m-\ln(K)}{s}\right) \\ &= K\,\Phi\left(\frac{\ln(S_0/K)+(\mu-\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right). \end{aligned}$$

□

It turns out that these results are everything we need in order to derive the Black-Scholes pricing formulas!

Martingales

A martingale is a process closely related to the random walk but slightly more general. A discrete-time martingale $\{Z_n\}_{n\geq 0}$ is a stochastic process such that:

$$\mathrm{E}\left(Z_{n+1} \mid Z_0, Z_1, \dots, Z_n\right) = Z_n.$$

Intuitively, the history of the process $\{Z_n\}$ is irrelevant to forecast Z_{n+1}. The current value of Z_n is the only thing that matters. A random walk is a martingale, but note that a martingale need not be a random walk.

For example, consider the process $\{Z_n\}$:

$$Z_{n+1} = Z_n \varepsilon_{n+1},$$

where $\{\varepsilon_n\}$ is an i.i.d. sequence such that $\mathrm{E}(\varepsilon_n) = 1$ for all $n \geq 0$. It is a martingale since:

$$\begin{aligned} \mathrm{E}(Z_{n+1} \mid Z_1, Z_2, \dots, Z_n) &= \mathrm{E}(Z_n \varepsilon_{n+1} \mid Z_n), \\ &= Z_n \, \mathrm{E}(\varepsilon_{n+1} \mid Z_n), \\ &= Z_n. \end{aligned}$$

Practice Problems

Solutions to all problems can be found at lorenzonaranjo.com/fin5241-fall25b.

Problem 1. Consider a stock whose price at time t is given by S_t and that follows a geometric Brownian motion (GBM). The expected return is 18% per year and the volatility is 32% per year. The current spot price is \$60.

a. Compute the expected price 9 months from now.
b. Compute the mean and standard deviation of the log-spot price 9 months from now.
c. Compute the 95% confidence interval of $\ln(S_T)$ 9-months from now, and report the corresponding values for S_T.

Problem 2. Consider a stock whose price at time t is given by S_t and that follows a GBM. The expected return is 11% per year and the volatility is 27% per year. The current spot price is \$60.

a. Compute the expected price of S_t 1 year from now.
b. Compute the expected price of $1/S_t$ 1 year from now.

Problem 3. Consider a stock whose price at time t is given by S_t and that follows a GBM. The expected return is 12% per year and the volatility is 35% per year. The current spot price is $55. Let $T = 18$ months.

a. Compute $\mathrm{E}(S_T)$.
b. Compute the mean and standard deviation of the log-spot price at T.
c. Find C such that $\mathrm{P}(S_T \leq C) = 0.01$.

Problem 4. Consider a stock whose price at time T is given by S_T and that follows a GBM, i.e.,

$$\ln(S_T) \sim \mathcal{N}(\ln(S_0) + (\mu - 0.5\sigma^2)T, \sigma^2 T).$$

The expected return is 12% per year and the volatility is 35% per year. The current spot price is $100.

a. Compute the expected price in 2 years from now.
b. Compute the mean and standard deviation of the log-spot price in 2 years from now.
c. Compute the probability that the spot price is less than $100 in 2 years from now.
d. Compute the probability that the spot price is greater than $120 in 2 years from now.

Problem 5. Suppose that the stock price follows a geometric Brownian motion (GBM) with drift μ and instantaneous volatility σ, i.e.,

$$\mathrm{d}S = \mu S\,\mathrm{d}t + \sigma S\,\mathrm{d}B.$$

Show that $Y = S^\alpha$ also follow a GBM and determine the drift and volatility as a function of μ, σ, and α.

Problem 6. Let S be the price of TESLA stock that follows a geometric Brownian motion such that

$$\mathrm{d}S = \mu S\,\mathrm{d}t + \sigma S\,\mathrm{d}B.$$

Your sales team would like to launch a new product called TESLA Quadro that tracks the price of TESLA to the power 4. In other words, the value of this instrument is given by $Y = S^4$. What is the process followed by Y?

Problem 7. GoingUp Corp. has been gaining a lot of attention in the media for its upside potential. Financial experts agree that the stock price follows a geometric Brownian motion with drift (μ) equal to 20% per year and volatility of price returns (σ) of 73% per year. The current stock price is \$220. Compute the probability that the stock price is greater than \$233 in 10 months from now.

Problem 8. You would like to invest in ZigZag Inc. but you are concerned that the stock price might go down. You have been studying the dynamics of the stock price and concluded that the stock follows a geometric Brownian motion with drift (μ) equal to 13% per year and volatility of price returns (σ) of 58% per year. The current stock price is \$118. Compute the probability that the stock price is less than \$98 in 12 months from now.

Problem 9. You are analyzing BMX stock. You believe that it is accurate to model the price evolution of the stock as a geometric Brownian motion. Using historical data, you estimate that the drift (μ) is 12.0% per year and the volatility of stock returns (σ) is 39% per year. The stock price just closed at \$331. Compute the expected stock price in 9 months from now.

Problem 10. Suppose that the stock price follows a geometric Brownian motion (GBM) with drift μ and instantaneous volatility σ. Show that $Y = Se^{-\mu t}$ also follow a GBM and determine the drift and volatility as a function of μ and σ.

Problem 11. Suppose that the stock price follows a geometric Brownian motion (GBM) with drift r and instantaneous volatility σ, where r is the risk-free rate. Consider the futures price of S at time t and expiring at T, given by $f = Se^{r(T-t)}$. Show that f has zero drift and hence is a martingale.

Problem 12. Suppose that the stock price follows a geometric Brownian motion (GBM) with drift $\mu = 10\%$ and instantaneous volatility $\sigma = 25\%$. Compute $\mathrm{E}(S_T \mathbf{1}_{\{S_T > K\}})$ and $\mathrm{E}(\mathbf{1}_{\{S_T > K\}}) = \mathrm{P}(S_T > K)$ if $S_0 = 100$, $K = 95$ and $T = 2$.