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The Black-Scholes Model

The Black-Scholes formula is one of the most celebrated results in finance. In this chapter we show how to replicate the payoff of a European call or put option written on a non-dividend paying stock by dynamically trading in the stock and a risk-free bond. The replication strategy is self-financing, and therefore determines the no-arbitrage price of the option.

A fundamental side-effect of the replication strategy is that the partial differential equation (PDE) that characterizes the price of the option does not depend on the real dynamics of the stock. We could obtain the same pricing equation by using any other risk-premia. This suggests a powerful idea to price the option. Let us assume that all investors are risk-neutral. If this was the case, the replication argument that gives the correct price of the option would still hold. However, in such a world, all assets should be priced by discounting their payoffs at the risk-free rate.

The risk-neutral approach provides us with a simpler way to derive the Black-Scholes formula. Since the risk-neutral dynamics of the non-dividend paying stock are driven by a GBM with drift equal to the risk-free rate, the stock price at maturity is log-normally distributed and allows us to apply the formulas for partial expectations in order to price the option.

The Replicating Portfolio Approach

In order to price a call or put option, we take the point of view of a trading desk that makes the market for option contracts. Their sales team just sold a European option H written on a non-dividend paying stock S with maturity T to a client. At this point, the traders of the desk are in charge of hedging the exposure of the short position.

The Hedge

Since the option depends on the stock, it makes sense to try to hedge the exposure by trading dynamically in the stock and a risk-free bond. Specifically, we will try to replicate the option by buying (or selling) N_S units of the stock and N_β units of a zero-coupon bond with face value K and maturity T, respectively, at each time $t \leq T$.[1] If we call V the value of such replicating portfolio, we have that at each time $0 \leq t \leq T$:

$$V_t = N_{S,t} S_t + N_{\beta,t} \beta_t.$$

In order to replicate the option, we want to make sure that the value of the portfolio at time $t = T$ equals the payoff of the derivative, that is:

$$V_T = F(S_T).$$

For example, if we consider a European call option then $F(S_T) = \max(S_T - K, 0)$, whereas for a European put option we have that $F(S_T) = \max(K - S_T, 0)$.

At time $t + \Delta t$, the value of the replicating portfolio is:

$$V_{t+\Delta t} = N_{S,t} S_{t+\Delta t} + N_{\beta,t} \beta_{t+\Delta t},$$

which implies that:

$$\Delta V_t = N_{S,t} \Delta S_t + N_{\beta,t} \Delta \beta_t.$$

As $\Delta t \to 0$, we have that:

$$\begin{aligned} \mathrm{d}V &= N_S \, \mathrm{d}S + N_\beta \, \mathrm{d}\beta = N_S \, \mathrm{d}S + N_\beta (r \beta \, \mathrm{d}t) \\ &= N_S \, \mathrm{d}S + (N_\beta \beta) r \, \mathrm{d}t, \end{aligned} \tag{1}$$

where in the second line we used the fact that $\mathrm{d}\beta = r\beta dt$.[2]

[1]To simplify notation, I will suppress the dependence on time whenever there is no ambiguity.
[2]Note that if $\beta = Ke^{-r(T-t)}$, then $\mathrm{d}\beta = rKe^{-r(T-t)} \, \mathrm{d}t = r\beta \, \mathrm{d}t$.

Now, the replication works in the following way. We determine first how many shares of the stock to buy or sell, depending on whether the option is a call or put. Then, given the number of shares that we need to hold and the value of the portfolio at time t, we see how much money we need to borrow or invest at the risk-free rate to keep our portfolio self-financing.

The amount invested in the risk-free bonds is such that $N_\beta \beta = V - N_S S$. This is a similar condition to the one imposed in classical portfolio theory where we keep the sum of the portfolio weights equal to one. Thus, if we replace $N_\beta \beta$ in (1) we get that

$$\mathrm{d}V = r(V - N_S S)\,\mathrm{d}t + N_S\,\mathrm{d}S. \tag{2}$$

Equation (2) captures the dynamics of the replicating portfolio needed to hedge the option that the trading desk just sold. As such, it represents the dynamics of the long position that the desk will hold to offset the risk of the short position. Thus, this is a classical long-short strategy. If the hedge is successful, the dynamics of both legs must also be the same.

The Exposure

As mentioned before, the sales team just sold a European option H written on a non-dividend paying stock S with maturity T to a client. Since our objective is to make sure that the value of the derivative is equal to the replicating portfolio, let us abuse notation and call for the moment also V the value of the derivative. If we assume that $V = V(S,t)$ is a smooth function of S and t, then Ito's Lemma implies that:

$$\begin{aligned}\mathrm{d}V &= \frac{\partial V}{\partial S}\,\mathrm{d}S + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}(\mathrm{d}S)^2 + \frac{\partial V}{\partial t}\,\mathrm{d}t \\ &= \frac{\partial V}{\partial S}\,\mathrm{d}S + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\,\mathrm{d}t + \frac{\partial V}{\partial t}\,\mathrm{d}t,\end{aligned}$$

where in the second we have used the fact that $(\mathrm{d}S)^2 = \sigma^2 S^2 \,\mathrm{d}t$. We can arrange the previous expression so that:

$$\mathrm{d}V = \left(\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t}\right)\mathrm{d}t + \frac{\partial V}{\partial S}\,\mathrm{d}S. \tag{3}$$

Equation (3) captures the dynamics of the short position.

Getting the Hedge to Work

We want to make sure that the hedge works so that the changes in value of the *option* equal the changes of the *replicating portfolio*. Therefore, replication will succeed if we can determine α such that both dynamics are the same:

$$\underbrace{\left(\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t}\right)\mathrm{d}t + \frac{\partial V}{\partial S}\,\mathrm{d}S}_{\text{Changes in the value of the option}} = \underbrace{r(V - N_S S)\,\mathrm{d}t + N_S\,\mathrm{d}S}_{\text{Changes in the replicating portfolio}} \tag{4}$$

Equation (4) shows that replication will indeed work if:

$$N_S = \frac{\partial V}{\partial S}. \tag{5}$$

This is a fundamental relationship in derivatives pricing. It states that the number of shares needed to replicate the derivative is equal its sensitivity to the underlying asset. The street name of this quantity is the Delta (Δ) of the derivative. Also, a by-product of choosing N_S to equal the Delta of the derivative is that it really does not matter what drift we have for the stock. We will use this fact in a moment to define the risk-neutral probabilities in continuous-time.

Second, it must be the case that:

$$\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t} = r\left(V - S\frac{\partial V}{\partial S}\right)$$

Therefore:

$$\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} + \frac{\partial V}{\partial t} - rV = 0, \tag{6}$$

subject to $V_T = F(S_T)$.

Equation (6) is the celebrated Black-Scholes partial differential equation (PDE), which allowed Fischer Black, Myron Scholes, and Robert Merton to develop their groundbreaking option pricing formula in 1973. This PDE characterizes the price of any European derivative written on a non-dividend paying stock as a function of the stock price and time.

The key insight from this PDE is that it does not depend on the expected return μ of the stock. This remarkable property arises because the replication strategy perfectly hedges away the risk associated with the stock's drift. The PDE only depends on observable or easily estimable quantities: the current stock price S, the risk-free rate r, and the volatility σ.

While solving PDEs in general is very challenging and often requires sophisticated mathematical methods such as Fourier transforms, we will take advantage of the drift-independence property to develop a more elegant and intuitive approach to price European call and put options: the risk-neutral pricing methodology.

The next chapter extends this analysis by deriving a more general PDE for derivatives on dividend-paying assets, where both the derivative and the underlying asset can pay a dividend yield.

The Risk-Neutral Pricing Approach

The replicating approach remains unaffected by the stock's drift. In fact, the drift could vary depending on the perspective of the investor considering the asset. Since the prior discussion does not address the drift or the investor's risk preference, we can assume that the investor performing the replication is *risk-neutral*. The risk attitude of the individual executing the replication does not influence *the logic* of the argument.

The Drift of the Stock is Irrelevant

In our model, the stock price follows a geometric Brownian motion such that:

$$\mathrm{d}S = \mu S dt + \sigma S \,\mathrm{d}B. \tag{7}$$

In equilibrium, if the stock returns co-vary positively with the returns of the market, as is typical for most stocks, the drift μ of the stock should exceed the risk-free rate r. Indeed, risk-averse investors would demand a risk premium to hold a risky asset that increases their exposure to the market.

However, equation (4) indicates that replication would succeed regardless of the value of μ. Since we can choose the parameter μ, let's explore the scenario where we set $\mu = r$, the risk-free rate. We can then rewrite (7) as:

$$\mathrm{d}S = rS \,\mathrm{d}t + \sigma S \,\mathrm{d}B^*. \tag{8}$$

Applying Ito's lemma to the derivative yields:

$$\begin{aligned} \mathrm{d}V &= \left(rS\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t} \right) \mathrm{d}t + \left(\sigma S \frac{\partial V}{\partial S} \right) \mathrm{d}B^* \\ &= rV dt + \left(\sigma S \frac{\partial V}{\partial S} \right) \mathrm{d}B^*. \end{aligned} \tag{9}$$

This equation (9) implies that if the expected return of the stock equals the risk-free rate, then the expected return of any derivative written on the stock will also be the risk-free rate.

This illustrates the characteristics of an economy comprised of *risk-neutral* investors.[3] In such a scenario, the expected return of any non-dividend paying asset aligns with the risk-free rate, as risk-neutral investors do not require a risk premium for holding risky assets in their portfolios.

[3] It is important to note that the probability distribution of the Brownian motion $\{B_t^*\}$ in a risk-neutral framework may not correspond to the physical measure observed in the real world, hence the asterisk.

With this understanding, the valuation of the derivative becomes straightforward. In a risk-neutral environment, the value of any risky asset is determined by its expected payoff discounted at the risk-free rate:

$$V = e^{-rT}\,\mathsf{E}^*(F(S_T)). \tag{10}$$

Thus, to price a European call or put option, we simply need to calculate the expectation of the final payoff under the assumption that the stock's drift equals r, and then discount this expected payoff at the risk-free rate.

Equation (10) provides an alternative method for computing the price of a derivative without necessitating the solution of the PDE defined in (6). However, the validity of the risk-neutral approach is grounded in our ability to replicate the derivative through trading in the stock and the risk-free bond.

Pricing a European Call Option

We can now use (10) to compute the premium of a European call option written on a non-dividend paying stock with maturity T and strike price K. The price of the call should then be

$$\begin{aligned}
C &= e^{-rT}\,\mathsf{E}^*\left((S_T - K)\mathbf{1}_{\{S_T > K\}}\right) \\
&= e^{-rT}\,\mathsf{E}^*\left(S_T\mathbf{1}_{\{S_T > K\}}\right) - e^{-rT}\,\mathsf{E}^*\left(K\mathbf{1}_{\{S_T > K\}}\right) \\
&= S\,\Phi(d_1) - Ke^{-rT}\,\Phi(d_2),
\end{aligned}$$

where

$$\begin{aligned}
d_1 &= \frac{\ln(S/K) + (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}, \\
d_2 &= d_1 - \sigma\sqrt{T}.
\end{aligned}$$

Note that in the third line we have used a property on partial expectations derived earlier. We now have a concrete valuation formula for the price of a European call!

Let's see how the call price varies with different values of the stock price.

The figure above shows that the call premium is an increasing function of the stock price, keeping everything else constant. This makes sense since for a given strike price, a higher

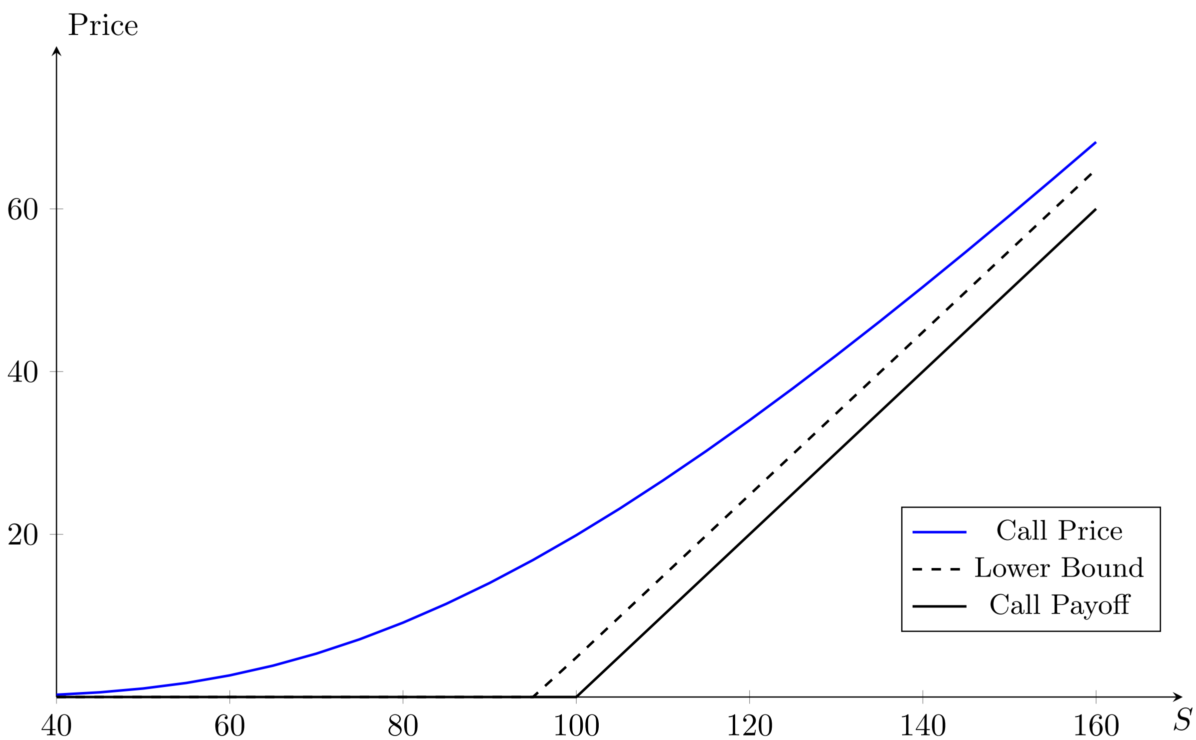

Figure 1: The figure plots the Black-Scholes call premium $C(S)$ if $r = 0.05$, $\sigma = 0.45$, $T = 1$ and $K = 100$. It also shows the call option payoff given by $\max(S - K, 0)$ and the lower bound for a European call given by $\max(S - Ke^{-rT}, 0)$.

stock price implies that the call is deeper in-the-money. In other words, the derivative of the call price with respect to the stock price must be positive. The graph also shows that the function is convex, meaning that the second derivative of the call price with respect to the stock price is also positive.

Call Delta

Practitioners usually call the number of shares required to make the replication work the call *delta*. Equation (5) shows that the number of shares N_S required to hedge the European call is the partial derivative of the call price with respect the current stock price. Now that we have an expression for the call price, we can compute the call delta explicitly.

> Call Delta
>
> In the Black-Scholes model, the delta of the European call is given by:
>
> $$\frac{\partial C}{\partial S} = \Phi(d_1). \tag{11}$$

Proof

We need to differentiate C with respect to S. Note that d_1 and d_2 are also functions of S:

$$\begin{aligned}
\frac{\partial C}{\partial S} &= \frac{\partial\,(S\,\Phi(d_1))}{\partial S} - Ke^{-rT}\frac{\partial\,\Phi(d_2)}{\partial S} \\
&= \Phi(d_1) + S\frac{\partial\,\Phi(d_1)}{\partial S} - Ke^{-rT}\frac{\partial\,\Phi(d_2)}{\partial S} \\
&= \Phi(d_1) + S\,\Phi'(d_1)\frac{\partial d_1}{\partial S} - Ke^{-rT}\,\Phi'(d_2)\frac{\partial d_2}{\partial S} \\
&= \Phi(d_1) + \underbrace{\left(S\,\Phi'(d_1) - Ke^{-rT}\,\Phi'(d_2)\right)}_{=0}\frac{1}{S\sigma\sqrt{T}} \\
&= \Phi(d_1),
\end{aligned}$$

where in the fourth line we used Equation (13). □

The call delta is then the slope coefficient of the call price with respect to the stock price. Because $\Phi(d_1) > 0$, the function $C(S)$ must be increasing in S.

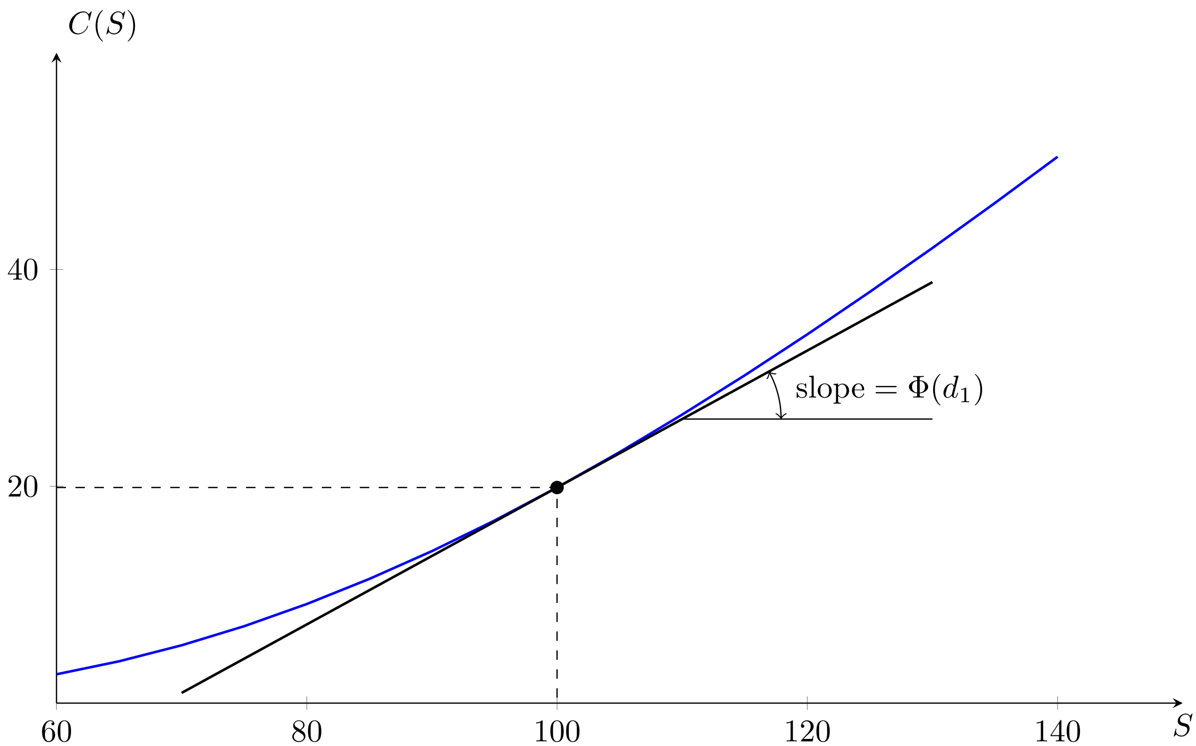

Figure 2: The figure plots the Black-Scholes call premium $C(S)$ where $r = 0.05$, $\sigma = 0.45$, $T = 1$ and $K = 100$, and shows the tangent line at $S = 100$ whose slope coefficient is the delta of the call given by $\Phi(d_1)$.

Therefore, the Delta of the call measures how sensitive is the call premium to small changes in the stock price. This is exactly why the trader needs to hold *Delta* shares of the stock to hedge the risk of the short call position.

Finally, we have that for a European call option:

$$C = N_S S + N_\beta \beta = S\,\Phi(d_1) - Ke^{-rT}\,\Phi(d_2),$$

which because of (11) implies that:

$$N_\beta = -\,\Phi(d_2).$$

Therefore, to replicate a European call option we need to go *long* $\Phi(d_1)$ shares of stock and *short* $\Phi(d_2)$ risk-free bonds with face value K and maturity T. The call can then be seen as a levered position in the underlying asset. Also, note that since $0 < \Phi(d_1) < 1$, the delta of the call for a non-dividend paying asset is bounded between 0 and 1. As we saw in the previous figure, for a given spot price, the delta of the call represents the slope coefficient of the tangency line at that point.

Pricing a European Put Option

Consider now a European put option with the same characteristics as the previous call. According to put-call parity, it must be the case that:

$$C - P = S - Ke^{-rT}$$

Hence,

$$\begin{aligned} P &= C - (S - Ke^{-rT}) \\ &= S\,\Phi(d_1) - Ke^{-rT}\,\Phi(d_2) - (S - Ke^{-rT}) \\ &= Ke^{-rT}(1 - \Phi(d_2)) - S(1 - \Phi(d_1)) \\ &= Ke^{-rT}\,\Phi(-d_2) - S\,\Phi(-d_1). \end{aligned}$$

Example 1. Consider a non-dividend paying stock that currently trades for \$100. The risk-free rate is 4% per year, continuously compounded and constant for all maturities. The instantaneous volatility of returns is 25% per year. Consider at-the-money call and put options written on the stock with maturity 9 months. Then,

$$\begin{aligned} d_1 &= \frac{\ln(100/100) + (0.04 + 0.5(0.25)^2)(0.75)}{0.25\sqrt{0.75}} = 0.2468, \\ d_2 &= 0.2468 - 0.25\sqrt{0.75} = 0.0303. \end{aligned}$$

Therefore, $\Phi(d_1) = 0.5975$ and $\Phi(d_2) = 0.5121$, which implies that:

$$\begin{aligned} C &= 100 \times 0.5975 - 100e^{-0.04\times 0.75} \times 0.5121 = \$10.05, \\ P &= 100e^{-0.04(0.75)} \times (1 - 0.5121) - 100 \times (1 - 0.5975) = \$7.10. \end{aligned}$$

$\square$

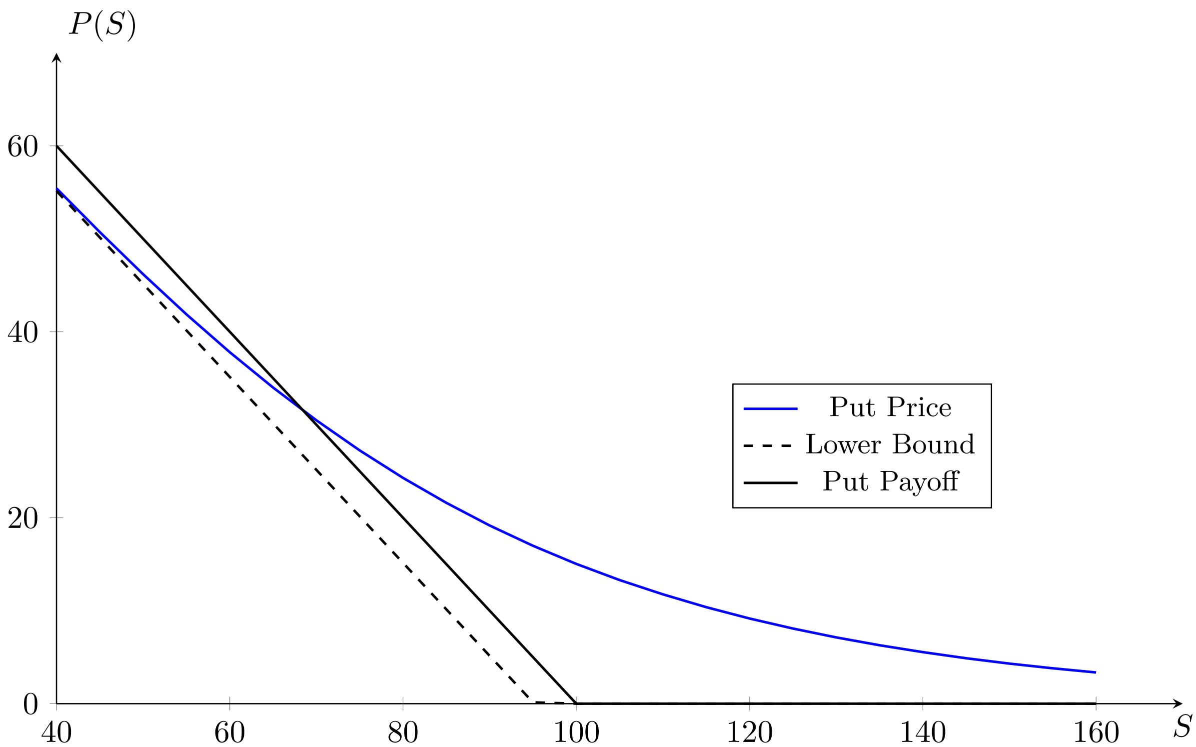

Figure 3: The figure plots the Black-Scholes put premium $P(S)$ if $r = 0.05, \sigma = 0.45, T = 1$ and $K = 100$. It also shows the put option payoff given by max$(K - S, 0)$ and the lower bound for a European put given by max$(Ke^{-rT} - S, 0)$.

Put Delta

We can use put-call parity to compute N_S for the put:

$$\begin{aligned} N_S = \frac{\partial P}{\partial S} &= \frac{\partial (C - S + Ke^{-rT})}{\partial S} \\ &= \Phi(d_1) - 1 = -\Phi(-d_1) < 0. \end{aligned}$$

The fact that we also have $P = N_S S + N_P P$ implies that:

$$N_Z = \Phi(-d_2) > 0.$$

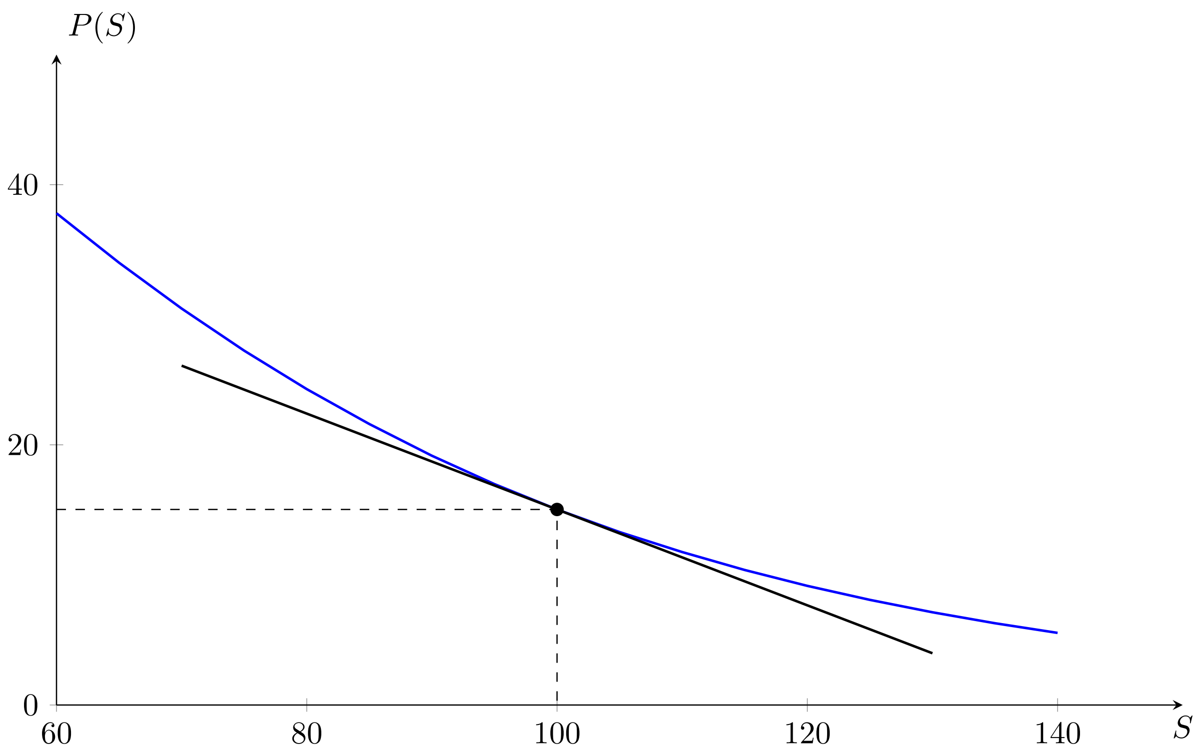

Figure 4: The figure plots the Black-Scholes put premium $P(S)$ where $r = 0.05, \sigma = 0.45, T = 1$ and $K = 100$, and shows the tangent line at $S = 100$ whose slope coefficient is the delta of the put given by $-\Phi(-d_1) = \Phi(d_1) - 1$.

Therefore, to replicate a European put option, we need to go *short* $\Phi(-d_1)$ shares of stock and *long* $\Phi(-d_2)$ risk-free bonds with face value K and maturity T.

Finishing In-The-Money

Remember that we showed that:

$$\mathsf{P}^*(S_T > T) = \mathsf{E}^*\left(\mathbf{1}_{\{S_T > K\}}\right) = \Phi(d_2),$$

which also implies

$$\mathsf{P}^*(S_T < K) = 1 - \mathsf{P}^*(S_T > K) = 1 - \Phi(d_2) = \Phi(-d_2).$$

Therefore, the risk-neutral probability that the call will expire in-the-money is equal to $\Phi(d_2)$ whereas the risk-neutral probability that the put finishes in-the-money is given by $\Phi(-d_2)$.

Summary

Black-Scholes Model for a Non-Dividend Paying Stock

Consider a non-dividend paying stock S that follows a GBM under the risk-neutral measure:

$$\mathrm{d}S = rS\,\mathrm{d}t + \sigma S\,\mathrm{d}B^*$$

The price of European call and put options with strike price K and time-to-maturity T are given by:

$$C = S\,\Phi(d_1) - Ke^{-rT}\,\Phi(d_2),$$
$$P = Ke^{-rT}\,\Phi(-d_2) - S_0\,\Phi(-d_1),$$

where

$$d_1 = \frac{\ln(S/K) + (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}},$$
$$d_2 = d_1 - \sigma\sqrt{T}.$$

Furthermore, the delta of the call is given by $\Phi(d_1)$ whereas the delta of the put is computed as $-\Phi(-d_1)$.
Finally, the risk-neutral probability that the call will expire in-the-money is equal to $\Phi(d_2)$ whereas the risk-neutral probability that the put finishes in-the-money is given by $\Phi(-d_2)$.

The Impact of Volatility

One of the most important determinants of option prices in the Black-Scholes model is volatility.

Black-Scholes Vega

For European call and put options we have that:

$$\frac{\partial C}{\partial \sigma} = \frac{\partial P}{\partial \sigma} = S\,\Phi'(d_1)\sqrt{T} > 0. \tag{12}$$

Proof

We start by differentiating C with respect to σ:

$$\begin{aligned}
\frac{\partial C}{\partial \sigma} &= S\frac{\partial\,\Phi(d_1)}{\partial \sigma} - Ke^{-rT}\frac{\partial\,\Phi(d_2)}{\partial \sigma} \\
&= S\,\Phi'(d_1)\frac{\partial d_1}{\partial \sigma} - Ke^{-rT}\,\Phi'(d_2)\frac{\partial d_2}{\partial \sigma} \\
&= S\,\Phi'(d_1)\frac{\partial d_1}{\partial \sigma} - Ke^{-rT}\,\Phi'(d_2)\frac{\partial (d_1 - \sigma\sqrt{T})}{\partial \sigma} \\
&= \underbrace{\left(S\,\Phi'(d_1) - Ke^{-rT}\,\Phi'(d_2)\right)}_{=0}\frac{\partial d_1}{\partial \sigma} + Ke^{-rT}\,\Phi'(d_2)\sqrt{T} \\
&= S\,\Phi'(d_1)\sqrt{T}.
\end{aligned}$$

Note that because of put-call parity we also have that:

$$
\begin{aligned}
C - P = S - Ke^{-rT} \Rightarrow \frac{\partial C}{\partial \sigma} - \frac{\partial P}{\partial \sigma} \\
= 0 \Rightarrow \frac{\partial P}{\partial \sigma} = \frac{\partial C}{\partial \sigma} \\
= S\,\Phi'(d_1)\sqrt{T}.
\end{aligned}
$$

$\square$

Hence, both European call and put options increase in value as volatility increases. Moreover, this also implies that there is a one-on-one relationship between option value and volatility, i.e., we can use volatility to quote prices and vice-versa. The volatility that matches the observed price of an option is called the *implied volatility*.

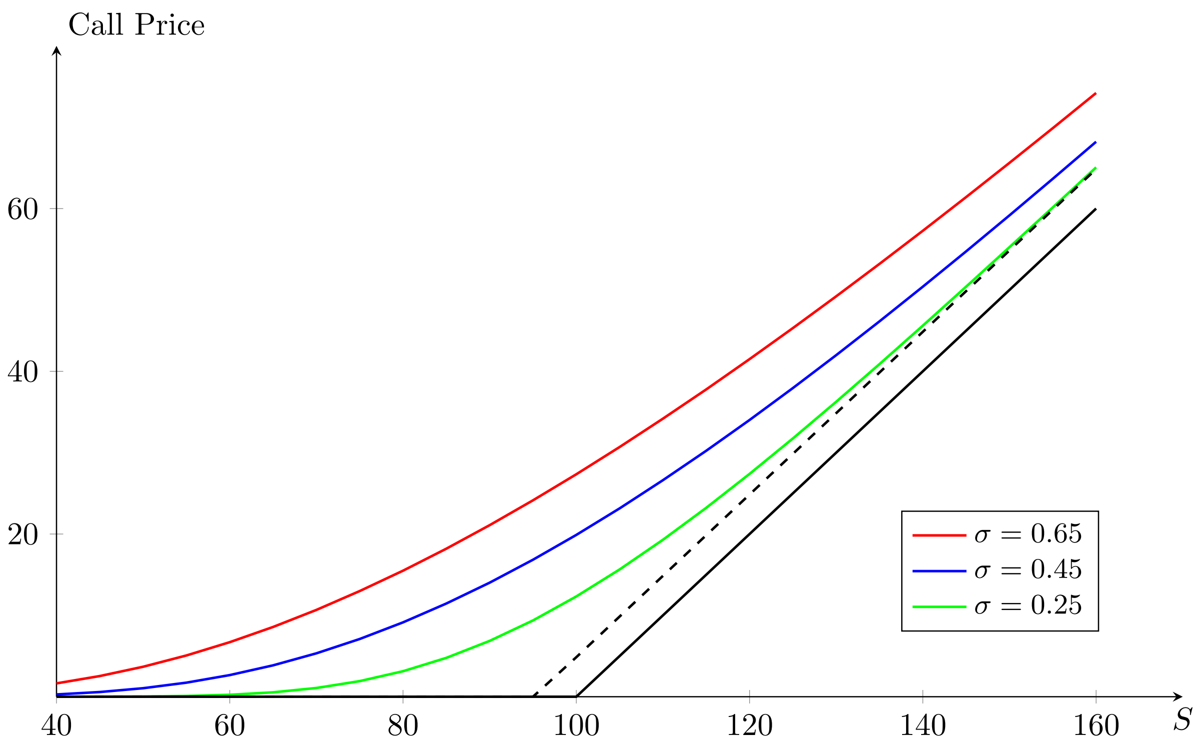

Figure 5: The figure shows the Black-Scholes call premium for different levels of volatility where $r = 0.05, T = 1$ and $K = 100$. The dashed line represents the lower bound for the European call and the solid black line is the call payoff at maturity.

Implied Volatility

Example 2. Consider a non-dividend paying stock that currently trades for \$100. The risk-free rate is 5% per year, continuously compounded and constant for all maturities. An ATM European call option written on the stock with maturity 12 months trades for \$16. We can check that $\sigma = 34.66\%$ prices the call correctly:

$$d_1 = \frac{\ln(100/100) + (0.05 + 0.5(0.3466)^2)(1)}{0.3466\sqrt{1}} = 0.3176,$$
$$d_2 = 0.3358 - 0.3466\sqrt{1} = -0.0290.$$

Therefore, $\Phi(d_1) = 0.6246$ and $\Phi(d_2) = 0.4884$, which implies that:

$$C = 100(0.6246) - 100e^{-0.05(1)}(0.4884) = \$16.00.$$

Therefore, a volatility of 34.66% per year gives a call price of \$16. □

How can we compute the implied volatility? Unfortunately, it is not possible to solve analytically for the implied volatility. For a call option, for example, it involves solving numerically for σ:

$$C_0 = C(\sigma_{imp})$$

Alternatively, we could tabulate the price of a call option for different values of σ (using the same parameters as the previous example):

σ	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40
C	5.28	6.80	8.59	10.45	12.34	14.23	16.13	18.02

We could see that $\sigma = 35\%$ gives a price of $\$16.13$ for the call, which is quite close to the true implied volatility of 34.66%.

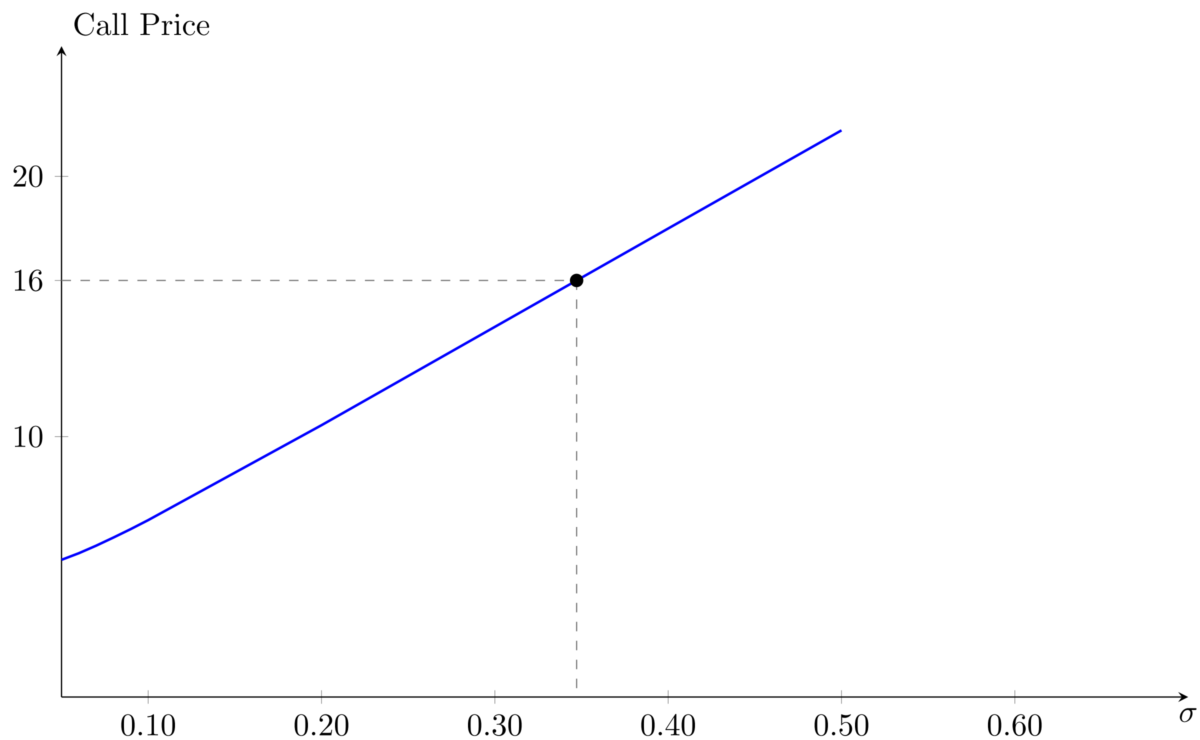

Figure 6: The figure shows the Black-Scholes call premium $C(\sigma)$ as a function of σ where $S = 100$, $r = 0.05$, $T = 1$ and $K = 100$. We can see that for $C = \$16$ the corresponding volatility is approximately 35%.

Appendix

In this section we present a useful result that will allow us to prove the formula for the Δ and Γ for European call and put options.

Indeed, we have that:

$$\begin{aligned}
(d_2)^2 &= (d_1 - \sigma\sqrt{T})^2 \\
&= (d_1)^2 - 2d_1\sigma\sqrt{T} + \sigma^2 T \\
&= (d_1)^2 - 2(\ln(S/K) + (r + \tfrac{1}{2}\sigma^2)T) + \sigma^2 T \\
&= (d_1)^2 - 2(\ln(S/K) + rT)
\end{aligned}$$

Hence,

$$\begin{aligned}
\Phi'(d_2) &= \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(d_2)^2} \\
&= \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left((d_1)^2 - 2(\ln(S/K) + rT)\right)} \\
&= \frac{S}{K} e^{rT}\, \Phi'(d_1)
\end{aligned}$$

which implies,

$$S\,\Phi'(d_1) = Ke^{-rT}\,\Phi'(d_2). \tag{13}$$

Practice Problems

Solutions to all problems can be found at lorenzonaranjo.com/fin5241-fall25b.

Problem 1. What is the price of a European put option on a non-dividend-paying stock when the stock price is \$69, the strike price is \$70, the risk-free interest rate is 5% per annum, the volatility is 35% per annum, and the time to maturity is six months?

Problem 2. A call option on a non-dividend-paying stock has a market price of \$2.61. The stock price is \$15, the exercise price is \$13, the time to maturity is three months, and the risk-free interest rate is 5% per annum. What is the implied volatility?

Problem 3. Consider an option on a non-dividend-paying stock when the stock price is \$30, the exercise price is \$29, the risk-free interest rate is 5%, the volatility is 25% per annum, and the time to maturity is four months.

a. What is the price of the option if it is a European call?
b. What is the price of the option if it is an American call?
c. What is the price of the option if it is a European put?
d. Verify that put-call parity holds.

Problem 4. Consider a European call option expiring in 6 months and with strike price equal to \$38 on a non-dividend paying stock that currently trades for \$40. Interestingly, the volatility of the stock is zero. If the risk-free rate is 6% per year with continuous compounding, what is the price of the option?

Problem 5. Consider a European call option expiring in 6 months and with strike price equal to \$42 on a non-dividend paying stock that currently trades for \$40. Interestingly, the volatility of the stock is zero. If the risk-free rate is 6% per year with continuous compounding, what is the price of the option?

Problem 6. Consider a European put option expiring in 9 months and strike price \$102 written on a non-dividend paying stock. The risk-free rate is 8% per year with continuous compounding and the stock price is \$100. What is the minimum price for the put that would allow you to compute its implied volatility?

Problem 7. Suppose that the sales team of a trading desk just sold a European call option contract, that is 100 European call options, to an important client. The contract is written on a non-dividend paying stock that trades for \$210, expires in two years and has a strike price of \$215. The risk-free rate is 6% per year with continuous compounding. A trader of the desk estimate that the volatility of the stock returns is 45% and expected to remain constant for the life of the contract.

a. How many shares of the stock does the trader need to buy/sell initially in order to hedge the exposure created by the sale of the contract?

b. How many risk-free bonds with face value \$215 and expiring in two years does the trader need to buy/sell in order to make sure that the strategy is self-financing?

Problem 8. According to the Black-Scholes model, what is the price of a European put option on a non-dividend-paying stock when the stock price is \$109, the strike price is \$106, the risk-free interest rate is 8% per year, the volatility is 29% per year, and the time to maturity is 14 months?

Problem 9. According to the Black-Scholes model, what is the price of a European call option on a non-dividend-paying stock when the stock price is \$85, the strike price is \$107, the risk-free interest rate is 7% per year, the volatility is 46% per year, and the time to maturity is 6 months?

Problem 10. According to the Black-Scholes model, what is the delta of a European put option on a non-dividend-paying stock when the stock price is \$92, the strike price is \$109, the risk-free interest rate is 6% per year, the volatility is 65% per year, and the time to maturity is 8 months?