Topics in Quantitative Finance — Prof. Lorenzo Naranjo
FIN 5018 — Fall 2025

Problem Set 2

Instructions: This problem set is due on 9/12 at 11:59 pm CST and is an individual assignment. All problems must be handwritten. Scan your work and submit a PDF file.

Problem 1. An investor lives in a world with four possible future states of the economy. In each state, the consumption level is different, and the investor's utility depends on consumption in each state. The states of the world are denoted as $\mathcal{S} = \{1, 2, 3, 4\}$, each occurring with equal probability.

The investor has a utility function $u(c) = \ln(c)$, and the stochastic discount factor $m(s)$ in state s is given by the formula:

$$m(s) = \beta \frac{u'(c_1(s))}{u'(c_0)}.$$

In this problem the investor can trade contingent claims written in each state of the world, so that the market is complete and there is a unique stochastic discount factor.

The current consumption is 110, and consumption next period in each state of the world is given by the vector

$$c_1 = \begin{pmatrix} 110 & 130 & 80 & 160 \end{pmatrix}^\top$$

The investor impatience parameter is $\beta = 0.95$.

a. Compute the stochastic discount factor for each state of the world.

b. Compute the price of a risk-free bond paying 100 next period.

c. The investor is considering purchasing a risky asset that pays state-contingent cash flows given by:

$$x = \begin{pmatrix} 10 & 25 & 5 & 30 \end{pmatrix}^{\top}$$

Calculate the price and expected return of the risky asset based on the stochastic discount factors you computed earlier.

d. Compute the price and expected of a call option with strike price 10 written on x and expiring next period.

e. Compute the risk-neutral probabilities in each state of the world, and verify that the price of the call can also be computed as the risk-neutral expectation of the cash flows discounted at the risk-free rate.

f. Compute the maximum Sharpe ratio that can be achieved in this economy.

g. Compute the prices π_i of Arrow-Debreu securities that pay one unit in state i and zero otherwise. For example, π_1 is the price of a security that pays $\begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix}^{\top}$, and π_3 is the price of a security that pays $\begin{pmatrix} 0 & 0 & 1 & 0 \end{pmatrix}^{\top}$. Using the Arrow-Debreu prices, show that there are no arbitrage opportunities in this economy.

h. Redo a. and b. assuming that $u(c) = -\frac{1}{c}$. Is the risk-free bond price higher or lower than in b.? Why?

Problem 2. Consider an investor with power utility so that $u'(c) = c^{-\gamma}$ and the stochastic discount factor is given by

$$m_{t+1} = \beta \left(\frac{c_{t+1}}{c_t} \right)^{-\gamma}.$$

Assume that $\Delta \ln c_{t+1} = \ln c_{t+1} - \ln c_t$ is conditionally normal with mean $\mathrm{E}_t(\Delta \ln c_{t+1})$ and variance $\sigma_t^2(\Delta \ln c_{t+1})$.

a. Show that $m_{t+1} = e^{-\delta - \gamma \Delta \ln c_{t+1}}$ if $\beta = e^{-\delta}$.

b. Using the fact that $\mathrm{E}(e^X) = e^{\mathrm{E}(X)+0.5\sigma^2(X)}$ if X is normally distributed with mean $\mathrm{E}(X)$ and variance $\sigma^2(X)$, show that

$$\mathrm{E}_t(m_{t+1}) = e^{-\delta-\gamma\,\mathrm{E}_t(\Delta \ln c_{t+1})+\frac{\gamma^2}{2}\sigma_t^2(\Delta \ln c_{t+1})}.$$

c. Show that the log real risk-free rate defined as $r_t^f = \ln(R_t^f)$ is given by

$$r_t^f = \delta + \gamma\,\mathrm{E}_t(\Delta \ln c_{t+1}) - \frac{\gamma^2}{2}\sigma_t^2(\Delta \ln c_{t+1}).$$

d. If future consumption is suddenly expected to be higher than anticipated, what should happen to the risk-free rate?

e. If real interest rates are low, what does this tell us about future consumption relative to current consumption?

f. Why if the volatility of future consumption increases then the real interest rate decreases? Explain the intuition behind the formula.

Problem 3. An investor seeks to maximize their lifetime utility, which is modeled using a CRRA additive utility function:

$$U(c_t, c_{t+1}, \dots) = \mathrm{E}_t \sum_{j=0}^{\infty} \beta^j \ln c_{t+j}.$$

Assuming consumption follows an endowment process, demonstrate that the price-to-consumption ratio of the endowment stream remains constant, irrespective of the stochastic properties of the consumption stream.