

Discount Factors in Continuous Time

Price Processes

In the following, we work in a probability space $(\Omega, \mathcal{F}, P)$. Uncertainty is driven by K independent Brownian motions such that

$$(d\mathbf{B})(d\mathbf{B})' = \mathbf{I}dt,$$

where $\mathbf{B} = \begin{pmatrix} B_1 & B_2 & \dots & B_K \end{pmatrix}'$ and $\mathbf{I}$ is a $K \times K$ identity matrix.

We can always generate correlated Brownian motions by combining two independent Brownian motions. For example, let $Z = \rho B_1 + \sqrt{1 - \rho^2} B_2$ where $|\rho| \leq 1$. Then, Z is a Brownian motion since $(dZ)^2 = \rho^2 dt + (1 - \rho^2) dt = dt$. Furthermore, Z and B_1 are instantaneously correlated since $(dZ)(dB_1) = \rho(dB_1)^2 = \rho dt$.

More generally, any Brownian motion Z correlated with $B_1, B_2, \dots, B_K$ is of the form

$$Z = \frac{1}{\sqrt{\sum_{k=1}^K a_k^2}} \sum_{k=1}^K a_k B_k,$$

where $a_1, a_2, \dots, a_K$ are real numbers, or in matrix notation

$$Z = \frac{1}{\sqrt{\mathbf{a}'\mathbf{a}}} \mathbf{a}'\mathbf{B}.$$

We model the price of risky assets as diffusions

$$\frac{dS}{S} = \mu_S(\cdot)dt + \sigma_S(\cdot)dB_S$$

where the drift $\mu(\cdot)$ and the volatility of returns $\sigma(\cdot)$ might depend on time t , uncertainty $\omega \in \Omega$, and potentially other state variables.

The total instantaneous return of an asset is given by

$$\frac{dS + Ddt}{S} = \frac{dS}{S} + \frac{D}{S}dt,$$

where $\frac{D}{S}$ is the dividend yield. The dividend yield of a stock determines the number of new shares that the dividend process generates. Unlike cash dividends, the dividend yield acts as if dividends are reinvested in the stock. Thus, we can always work with dividend-reinvested assets instead. To see this, let

$$\frac{dX}{X} = \frac{D}{S}dt. \quad (1)$$

Here X represents the number of new shares accruing to the owner of the stock determined by the dividend yield. Note that

$$X_t = X_0 \exp\left(\int_0^t \frac{D_u}{S_u} du\right), \quad (2)$$

so the total number of shares grows exponentially with a growth rate equal to the dividend yield of the asset. In other words, X_t keeps track of the total number of shares at each point in time.

The dividend-reinvested asset price is then

$$P = XS,$$

where P denotes the total value of this investment given by the number of shares times the price per share. We can find the dynamics of P by applying Ito's lemma:

$$\frac{dP}{P} = \frac{dS}{S} + \frac{dX}{X} = \frac{dS}{S} + \frac{D}{S}dt.$$

Not surprisingly, the dynamics of P are characterized by capital gains and a dividend yield.

We will assume that there is a risk-free rate of return r . We do not always assume that r is constant, as it could depend on time t , ω or other state variables. We assume that there is a money-market account β that earns a risk-free rate. If we start with β_0 in the account, we must have that

$$\frac{d\beta}{\beta} = r dt. \quad (3)$$

We can solve for β to find,

$$\beta_t = \beta_0 \exp\left(\int_0^t r_s ds\right). \quad (4)$$

As mentioned before, we do not assume that r is constant. In many applications, the risk-free rate follows a diffusion such that

$$dr = \mu_r(\cdot)dt + \sigma_r(\cdot)dB_r.$$

From Discrete to Continuous Time

We want to derive a stochastic discount factor that works discounting risky cash flows in continuous time. It will be easier to derive the transition if we can abstract from dividends. However, we cannot just make dividends disappear since the fundamental value of any asset is the present value of all cash flow payments during the life of the asset; otherwise, the asset would be a bubble.

Thus, let's proceed as before and build a dividend-reinvested asset in discrete time. To do this, we will use a dividend yield q that is known at time t and determines how many new shares of the asset we get next period. If the asset price at time 0 is S_0 , then at time 1 the asset price will be S_1 and we will have q_0 more shares of the asset, making our total investment worth $P_1 = (1 + q_0)S_1$. If reinvest the dividends again, at time 2 the value of our investment will be $P_2 = (1 + q_0)(1 + q_1)S_2$. We can continue in this way to find

$$P_{t+1} = X_{t+1}S_{t+1},$$

where we define $X_{t+1} = \prod_{i=0}^t (1 + q_i)$. This definition for X_{t+1} is the discrete-time equivalent of equation (2) in continuous time.

Now, the pricing equation implies that

$$S_t = E_t m_{t+1} (1 + q_t) S_{t+1}.$$

Multiplying both sides by $\prod_{i=0}^{t-1} (1 + q_i)$ we find

$$P_t = E_t m_{t+1} P_{t+1}. \quad (5)$$

Thus, we can do all the math by assuming that we work with a dividend-reinvested asset.

The previous expression is correct in discrete time but it is not the best way to work in continuous time. First, assume that the stochastic discount factor we use is such that $m > 0$ a.s. Then, take $\Lambda_0 > 0$ and define $\Lambda_1 = m_1 \Lambda_0$, $\Lambda_2 = m_2 \Lambda_1$, so that $\Lambda_{t+1} = m_{t+1} \Lambda_t > 0$ for all $t \geq 0$.

We can now re-write equation (5) as

$$\Lambda_t P_t = E_t \Lambda_{t+1} P_{t+1}. \quad (6)$$

Before passing to continuous time, note that we can recursively apply the previous expression to P_{t+2} , P_{t+3} , and so forth to find

$$\Lambda_t P_t = E_t \Lambda_T P_T,$$

for any $T = t + n > t$. The process (ΛP) is therefore a martingale. We need to be careful, though, as we do not want the price process P to be a bubble. In terms of the traded asset S we must have that

$$\Lambda_t S_t = E_t \sum_{i=1}^n \Lambda_{t+i} D_{t+i} + \Lambda_{t+n} S_{t+n}.$$

The no-bubbles or transversality condition then implies that

$$\lim_{n \rightarrow \infty} E_t \Lambda_{t+n} S_{t+n} = 0.$$

In continuous time we do not have to wait forever for bubbles to appear since in any given time interval we have an infinite number of transactions that can occur. Thus, we will

have to be careful about what type of price processes we can admit.

Coming back to passing to continuous time, let's denote the time interval by Δt so that equation (6) becomes

$$E_t(\Lambda_{t+\Delta}P_{t+\Delta} - \Lambda_tP_t) = 0.$$

If we now let $\Delta t \rightarrow 0$, the previous expression implies

$$E_t d(\Lambda_t P_t) = 0.$$

We typically drop the time subscripts when there is not confusion of doing so and just write

$$E d(\Lambda P) = 0. \quad (7)$$

The previous expression asserts that the discounted dividend-reinvested price process is a local martingale. We will discuss later situations in which this local martingale is in fact a martingale.

Since $\Lambda > 0$, we have that

$$\begin{aligned} \frac{d(\Lambda P)}{\Lambda P} &= \frac{dP}{P} + \frac{d\Lambda}{\Lambda} + \frac{d\Lambda}{\Lambda} \frac{dP}{P} \\ &= \frac{dS}{S} + \frac{d\Lambda}{\Lambda} + \frac{d\Lambda}{\Lambda} \frac{dS}{S} + \frac{D}{S} dt \\ &= \frac{d(\Lambda S)}{\Lambda S} + \frac{D}{S} dt. \end{aligned}$$

Thus, $E d(\Lambda P) = 0$ is equivalent to

$$E d(\Lambda S) + \Lambda D dt = 0. \quad (8)$$

Equation (8) is an alternative to equation (7) which makes explicit the dividend process in pricing the asset. Both equations are the continuous time counterparts to $p = E(mx)$ in discrete time, which in disguise is also saying that the discounted dividend-reinvested price process is a local martingale.

An SDF in Continuous Time

Let's start computing the discounted process for the money market account β defined earlier. Remember that

$$\frac{d\beta}{\beta} = r dt.$$

Thus,

$$d(\Lambda\beta) = \Lambda d\beta + \beta d\Lambda.$$

The pricing equation (8) implies that $E d(\Lambda\beta) = 0$, so that

$$E(\Lambda d\beta + \beta d\Lambda) = 0,$$

or

$$E\left(\frac{d\Lambda}{\Lambda}\right) = -E\left(\frac{d\beta}{\beta}\right) = -r dt.$$

Thus, the drift of the SDF in continuous time determines the equilibrium continuously-compounded risk-free rate. Remember that r need not be deterministic but just an adapted process to the filtration of the probability space.

Applying Ito's lemma now to ΛS we find that

$$d(\Lambda S) = S d\Lambda + \Lambda dS + d\Lambda dS,$$

or

$$\frac{d(\Lambda S)}{\Lambda S} = \frac{d\Lambda}{\Lambda} + \frac{dS}{S} + \frac{d\Lambda}{\Lambda} \frac{dS}{S}.$$

Taking expectations both sides, equation (8) implies that

$$E\left(\frac{dS}{S}\right) + \frac{D}{S} dt = r dt - \left(\frac{d\Lambda}{\Lambda}\right)\left(\frac{dS}{S}\right).$$

Property 1. Consider an asset S that follows a diffusion

$$\frac{dS}{S} = \mu dt + \sigma dB.$$

If the asset pays a dividend yield $q = D/S$, and there are no arbitrage opportunities, it must be the case that

$$(\mu + q - r)dt = -\left(\frac{d\Lambda}{\Lambda}\right)\left(\frac{dS}{S}\right). \quad (9)$$

In words, the risk-premium of the asset equals minus the covariance of the SDF and the asset's returns.

Back to Consumption

We can always find the stochastic discount factor from the marginal utility of consumption. Under additive utility, the stochastic discount factor takes the form $\Lambda = e^{-\delta t}u'(c)$. This formulation is useful if we want to understand the link between marginal utility of consumption and discount factors or risk-neutral probabilities.

Applying Ito's lemma to this particular Λ we find

$$\begin{aligned} d\Lambda &= \frac{\partial \Lambda}{\partial c}dc + \frac{1}{2}\frac{\partial^2 \Lambda}{\partial c^2}(dc)^2 + \frac{\partial \Lambda}{\partial t}dt \\ &= e^{-\delta t}u''(c)dc + \frac{1}{2}e^{-\delta t}u'''(c)(dc)^2 - \delta e^{-\delta t}u'(c)dt, \end{aligned}$$

or

$$\frac{d\Lambda}{\Lambda} = -\delta dt + \frac{1}{2}\frac{c^2 u'''(c)}{u'(c)}\left(\frac{dc}{c}\right)^2 + \frac{cu''(c)}{u'(c)}\frac{dc}{c}.$$

For power utility we have that

$$\frac{d\Lambda}{\Lambda} = -\delta dt + \frac{1}{2}\gamma(\gamma + 1)\left(\frac{dc}{c}\right)^2 - \gamma\frac{dc}{c}.$$

Let's write

$$\frac{dc}{c} = \mu_c dt + \sigma_c dB_c.$$

Assuming power utility, we have that

$$\frac{d\Lambda}{\Lambda} = \left(-\delta + \frac{1}{2}\gamma(\gamma + 1)\sigma_c^2 - \gamma\mu_c \right) dt - \gamma\sigma_c dB_c. \quad (10)$$

We can now recover the risk-free rate dynamics in terms of consumption growth dynamics using

$$r = -\frac{1}{dt} E \left(\frac{d\Lambda}{\Lambda} \right).$$

The model implies that the instantaneous risk-free rate is given by minus the drift of $d\Lambda/\Lambda$,

$$r = \delta + \gamma\mu_c - \frac{1}{2}\gamma(\gamma + 1)\sigma_c^2.$$

The expression is very intuitive and has important implications. First, real interest rates are high when impatience (δ) is high since more impatient investors will require a high interest rate to save. Furthermore, interest rates are high when expected consumption growth (μ_c) is high. Indeed, if agents expect consumption to go up, they need to save less, pushing bond prices down. Finally, interest rates are high when volatility of future consumption growth (σ_c) is low. This phenomenon is usually called **precautionary savings**. If agents are less afraid of future consumption growth, they bid bond prices down pushing interest rates up.

We can also use this consumption-based asset pricing model to understand what is called the **equity premium puzzle**. Consider an asset paying a dividend flow Ddt and following a diffusion

$$\frac{dS}{S} = \mu_S dt + \sigma_S dB_S$$

such that $(dB_S)(dB_c) = \rho dt$. Equations (9) and (10) imply

$$\mu_S + D/S - r = \gamma\rho\sigma_c\sigma_S$$

In this simple asset pricing model with power utility, the risk premium of any risky asset is higher when risk aversion (γ) is high and/or the covariance of asset returns and consumption growth is high.

Since $|\rho| \leq 1$, the previous expression implies

$$\left| \frac{\mu_S + D/S - r}{\sigma_S} \right| \leq \gamma \sigma_c.$$

In the data, the Sharpe ratio of the market is around 0.5 whereas the standard deviation of consumption growth is around 0.01. We need a RRA coefficient of at least 50 to explain the risk-premium of the market! To solve this paradox, researchers have introduced preferences that generate a more volatile stochastic discount factor, such as recursive Epstein-Zin preferences or habits.

Generic SDFs in Continuous Time

There are $N \leq K$ securities whose price process follow a diffusion

$$\frac{dS_i}{S_i} = \mu_i dt + \sigma_i d\mathbf{B}, \quad (11)$$

where σ_i is a $K \times 1$ vector. Each security pays continuously a dividend yield $q_i dt$.

Define

$$\frac{d\mathbf{S}}{\mathbf{S}} = \begin{pmatrix} \frac{dS_1}{S_1} & \frac{dS_2}{S_2} & \dots & \frac{dS_N}{S_N} \end{pmatrix}'$$

and denote by σ the $N \times K$ matrix whose rows are given by σ_i defined in (11). We have that

$$\frac{d\mathbf{S}}{\mathbf{S}} = \boldsymbol{\mu} dt + \sigma d\mathbf{B},$$

implying

$$\left(\frac{d\mathbf{S}}{\mathbf{S}} \right) \left(\frac{d\mathbf{S}}{\mathbf{S}} \right)' = \sigma \sigma' dt.$$

The $N \times N$ matrix $\sigma \sigma'$ determines the instantaneous covariance of returns. We can verify

$$\frac{d\Lambda}{\Lambda} = -r dt - (\boldsymbol{\mu} + \mathbf{q} - r\mathbf{1})' (\sigma \sigma')^{-1} \sigma d\mathbf{B}$$

is an SDF that prices the N original assets correctly. In the expression $\mathbf{q}$ denotes the vector of dividend yields

$$\mathbf{q} = \left(\frac{D_1}{S_1}, \frac{D_2}{S_2}, \dots, \frac{D_N}{S_N} \right)'.$$

The Intertemporal CAPM

Let $V(W, \mathbf{z})$ denote the value function that the investor maximizes at each time. The level of wealth W is of course a state variable. The value function also depends on the risks to which all the assets are exposed, represented by $\mathbf{z}$. In a more general setup, the marginal utility of consumption is the same as the marginal utility of wealth, i.e.

$$u'(c) = V_W.$$

The marginal value of any dollar must be the same in any use! We can then write the SDF as

$$\Lambda = e^{-\delta t} V_W(W, \mathbf{z}).$$

Applying Ito's lemma to Λ we find that:

$$\frac{d\Lambda}{\Lambda} = (\cdot)dt + \frac{W V_{WW}(W, \mathbf{z})}{V_W(W, \mathbf{z})} \frac{dW}{W} + \frac{V_{W\mathbf{z}'}(W, \mathbf{z})}{V_W(W, \mathbf{z})} d\mathbf{z}. \quad (12)$$

Thus

$$\mathbb{E} \left(\frac{dS}{S} \right) + \frac{D}{S} dt - r dt = rra \frac{dW}{W} \frac{dS}{S} - \frac{V_{W\mathbf{z}'}(W, \mathbf{z})}{V_W(W, \mathbf{z})} \left(d\mathbf{z} \frac{dS}{S} \right).$$