Interest Rate Risk Management

Bond Price Sensitivity

The term structure of interest rates changes over time, and this of course affects the prices of fixed-income securities. We call *interest rate sensitivity* how much bond prices change when interest rates change.

The figure below shows how the price of four different bonds change when the YTM changes. We say that a bond is more price sensitive than another if the percentage price change is larger for an equal change in YTM.

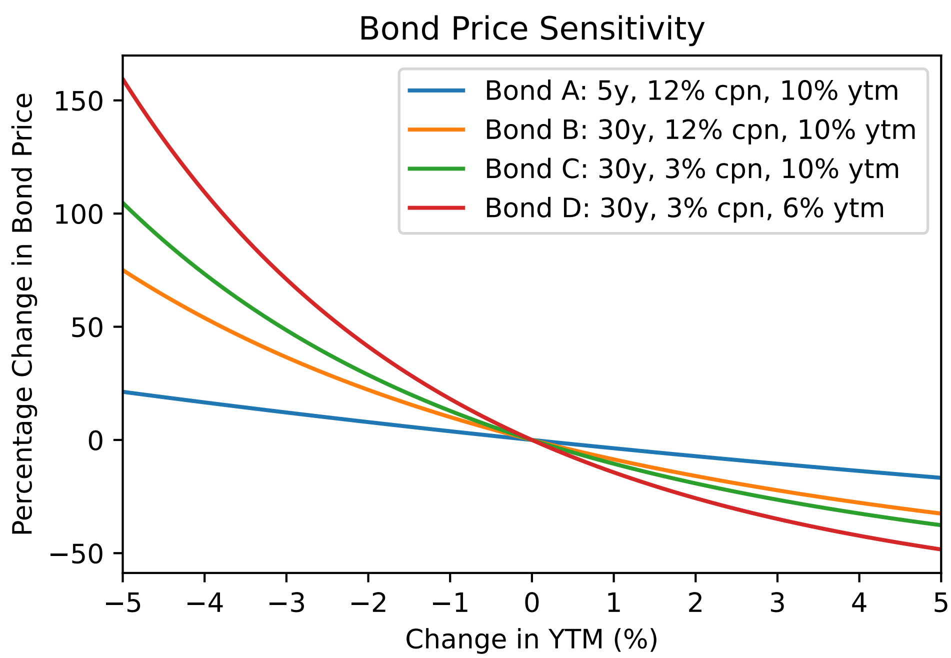

Figure 1: The figure shows the price sensitivity of different bonds vs. changes in YTM.

We already saw that bond prices and yields are inversely related, although not in a linear fashion. Indeed, an increase in a bond's yield to maturity results in a smaller change in price than a decrease in the yield of equal magnitude.

Comparing bonds A and B that have the same coupon rate and the same initial YTM, we can see that long-term bonds are more price sensitive than short-term bonds. Comparing bonds B and C, we can see that for the same maturity the price sensitivity is inversely related to the bond's coupon rate. Finally, comparing bonds C and D we can see that the price sensitivity of a bond is inversely related to the yield to maturity at which the bond is selling.

Duration

It turns out that the right way to measure bond price sensitivity is to compute it, and that leads to the concept of *duration*. To make the analysis general, consider a series of cash flows C_t paid at the end of each year $t = 1, 2, \dots, T$. Let us assume for the moment that the yield curve is flat for all maturities so that all cash flows are discounted at the same rate y expressed per year with annual compounding.

$$V(y) = \frac{C_1}{(1+y)^1} + \frac{C_2}{(1+y)^2} + \cdots + \frac{C_T}{(1+y)^T}$$

In the analysis that follows we will keep the maturity T of the cash flows constant. That is, we are trying to measure the sensitivity of V with respect to an unforeseen and instantaneous change in y. We then have that:

$$\frac{dV}{dy} = -1 \times \frac{C_1}{(1+y)^2} - 2 \times \frac{C_2}{(1+y)^3} - \cdots - T \times \frac{C_T}{(1+y)^{T+1}}.$$

The previous expression can be rewritten as:

$$\frac{dV}{V} = -\frac{D}{1+y}dy,$$

where

$$D = 1 \times w_1 + 2 \times w_2 + \cdots + T \times w_T,$$

and

$$w_t = \frac{C_t/(1+y)^t}{V}.$$

The term D is usually called the *Macaulay duration* of the cash flows. The effective price sensitivity of the cash flows is given by:

$$D_{\text{mod}} = \frac{D}{1+y}$$

and is usually called the *modified duration* of the cash flows.

The figure below shows the duration of different bonds as a function of their time to maturity.

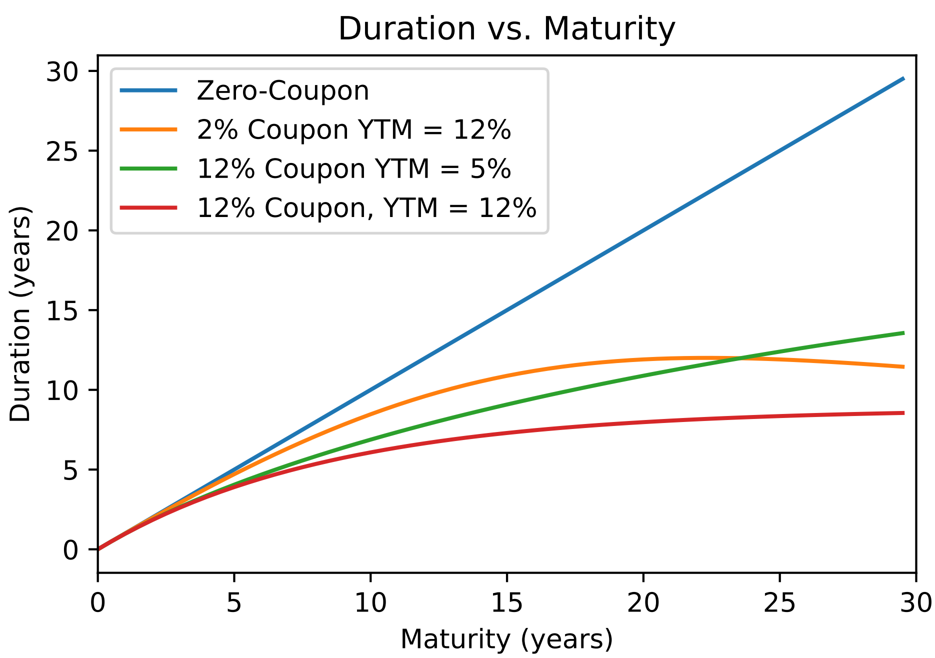

Figure 2: The figure shows the duration of different bonds vs. their time to maturity.

The figure displays interesting properties of the duration of a bond:

- The duration of a zero-coupon bond equals its maturity. Indeed, if there are no coupons there is only one weight that accounts for 100% of the duration at maturity.

- Holding maturity constant, a bond's duration is higher when the coupon rate is lower. A lower coupon rate means that more weight will go to the principal, increasing its duration.
- Holding the coupon rate constant, a bond's duration generally increases with its time to maturity. In some cases, though, especially when the YTM is very high compared to the coupon rate, the weight of the face value might decrease with maturity making the weight of the other coupons to increase.
- Holding other factors constant, the duration of a coupon bond is higher when the bond's yield to maturity is lower. This is because a lower YTM decreases the discounting of the face value, which then contributes to increase the duration.

Example 1 (Duration of a Perpetuity)**.** Consider a perpetuity that pays an annual coupon C when the discount rate is y expressed per year with annual compounding. The value of the perpetuity is:

$$V = \frac{C}{y} \Rightarrow \frac{dV}{dy} = -\frac{C}{y^2}.$$

Thus,

$$\frac{dV}{V} = -\frac{1}{y}dy = -\frac{D}{1+y}dy$$

The modified duration of the perpetuity is then $1/y$ and its Macaulay duration $(1+y)/y$.

Modified duration is usually used to approximate the change in value of a bond or portfolio for a change in yields.

Example 2 (Change in Value of a Bond)**.** Consider a 4-year annual paying coupon bond with face value $1,000, a coupon rate of 8% and a YTM of 10% per year with annual compounding. The duration of the bond can be calculated as follows.

Maturity	Cash flow	Discounted cash flow	Weight
1	80	72.73	7.77%
2	80	66.12	7.06%
3	80	60.11	6.42%
4	1080	737.65	78.76%

Maturity	Cash flow	Discounted cash flow	Weight
Total		936.60	100.00%

The Macaulay duration is then

$$D = 0.0777 \times 1 + 0.0706 \times 2 + 0.0642 \times 3 + 0.7876 \times 4 = 3.56.$$

If the YTM changes from 10% to 10.50%, the percentage change in price will be:

$$\begin{aligned}\frac{\Delta V}{V} &\approx -\frac{3.56}{1.10}\Delta y \\ &= -\frac{3.56}{1.10} \times 0.0050 \\ &= -1.62\%.\end{aligned}$$

Convexity

The sensitivity of price with respect to yield is approximated by a linear function when using duration. The relation is really non-linear, in particular, it is convex. The convexity of a bond is the curvature of its price-yield relationship. The relative price response to a yield change can be better approximated using convexity:

$$\frac{\Delta V}{V} \approx -\frac{D}{1+y}\Delta y + \frac{1}{2}\text{convexity}(\Delta y)^2.$$

where

$$\text{convexity} = \frac{d^2V}{dy^2}\frac{1}{V} = \sum_{t=1}^{T} \frac{t+t^2}{(1+y)^2} w_t,$$

and the weight is computed as for duration.

Example 3 (Change in Value of a Bond)**.** Using the data of Example 2, we computed $w_1 = 0.0777, w_2 = 0.0706, w_3 = 0.0642, w_4 = 0.7876$ and $D = 3.56$. The convexity of a 4-year annual paying coupon bond with face value \$1,000, a coupon rate of 8% and a YTM of 10% per year with annual compounding is:

$$\text{convexity} = \frac{1+1^2}{1.10^2}w_1 + \frac{2+2^2}{1.10^2}w_2 + \frac{3+3^2}{1.10^2}w_3 + \frac{4+4^2}{1.10^2}w_4 = 14.13.$$

Adjusting for convexity, if the YTM changes from 10% to 10.50%, the percentage change in price will be approximately equal to:

$$\begin{aligned}\frac{\Delta V}{V} &\approx -\frac{3.56}{1.10}\Delta y + \frac{1}{2}14.13(\Delta y)^2 \\ &= -\frac{3.56}{1.10} \times 0.0050 + \frac{1}{2}14.13(0.0050)^2 \\ &= -1.60\%.\end{aligned}$$

When yields decline, the price increase in the bond is underestimated by the simple duration formula. A convexity term corrects the problem. The more convex a bond, the greater the expected price increase for a given decrease in yield and the smaller the expected price decrease. If interest rates are volatile, this is an attractive asymmetry. Investors will have to pay higher prices (accept lower yields) for bonds with more convexity.

Managing Interest Rate Risk Exposure

Investors and financial institutions are subject to interest-rate risk, for instance,

- homeowner: mortgage payments (ARM)
- bank: short-term deposits and long-term loans
- pension fund: owns bonds and must pay retirees

A change in the interest rate results in:

- price risk
- re-investment risk

We will try to construct a portfolio which is insensitive to interest-rate changes.

Immunization

Duration matching or *immunization* means to make the duration of assets and liabilities equal. Then, the sensitivity to interest-rate changes is:

$$\Delta V \approx \frac{D^{\text{assets}}}{1+y} V^{\text{assets}} \Delta y - \frac{D^{\text{liabilities}}}{1+y} V^{\text{liabilities}} \Delta y = 0$$

If this is the case, interest rate changes makes the values of assets and liabilities change by the same amount: The portfolio is *immunized*.

Example 4 (The Savings & Loans Crisis)**.** Michael Lewis in his book Liar's Poker described Savings & Loans (S&L) members as part of the 3-6-3 club: Borrow money at 3 percent, lend it out at 6 percent, and be on the golf course every afternoon by three o'clock.

S&L's had predominantly short-term deposits (short duration liabilities) and long-term mortgage loans (long duration assets). William Poole, former president FRB St. Louis, declared:

The decline of the savings institutions [in the 80s] was a consequence of rising nominal interest rates combined with duration mismatch.

Example 5 (Pension Fund)**.** A company's pension fund had liabilities with duration of about 15 years assets (bonds) with duration of about 5 years. This is a duration mismatch.

- Price risk:
  - When the interest rate falls, the value of the bonds increases, but the present value of the liabilities increases more.
- Reinvestment risk:
  - At the new interest rate, the assets could not be reinvested to make the future payments.

To make things concrete, assume that the pension fund has to pay $100 million in 15 years, and that the current interest rate is 6% per year with annual compounding for all maturities.

Suppose that the pension fund wants to invest in 1-year and 30-year zero-coupon bonds. Remember that the duration of a zero coupon bond is equal to its maturity. If we denote by w_1 the percentage invested in 1-year bonds and by $w_{30} = 1 - w1$ the percentage invested in 30-year bonds, it must be the case that:

$$1 \times w_1 + 30 \times (1 - w_1) = 15.$$

which implies that

$$w_1 = \frac{30 - 15}{30 - 1} = \frac{15}{29} \quad \text{and} \quad w_{30} = \frac{14}{29}.$$

Since the present value of the liabilities is $100/1.06^{15} = \$41.727$ million, the fund needs to invest $(15/29) \times 41.727 = \$21.583$ million in 1-year bonds and $41.727 - 21.583 = \$20.144$ million in 30-year bonds to immunize its portfolio.

Problems with Immunization

Immunization in general requires the strategy to be rebalanced. As was shown previously, it is an approximation that assumes:

- A flat term structure of interest
- Only risk of changes in the level of interest, but not in the slope of the term-structure or other types of shape changes
- Small interest rate changes: improve duration matching by also matching convexity