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Optimal Capital Allocation

If passive investors possess the same information, they should agree on which combination of risky assets provides the best trade off between risk and return. Nevertheless, some investors may think this optimal portfolio of risky assets carries too much risk. Others may think the opposite.

It is possible to reduce or increase the risk of a portfolio by investing or borrowing a risk-free asset. The *capital allocation* decision is then about how much to invest in this well-diversified portfolio of risky investments and how much to allocate to a risk-free asset.

The solution to the capital allocation problem has two dimensions:

- Model what is available to invest,
- Understand what investors want.

The first point requires us to determine the *investment opportunity set*. For example, we would love to be able to invest our savings in a risk-free deposit account that yields 20% per year. Unfortunately, such an account does not exist. Combining a risky asset with the risk-free rate generates an investment set called the *capital allocation line*.

The second point has to do with how investors feel when taking risks. Investors dislike risk but like returns. In finance and economics, we capture these two opposite effects using a *utility function*.

The Capital Allocation Line

We will now analyze the investment opportunity set generated by a risky asset Q and a risk-free asset. We denote by r_f the risk-free rate of return. The expected return of the risky asset is denoted by μ_Q whereas the standard deviation or volatility of fund returns is denoted by σ_Q.

A portfolio P that invest w in Q and $1-w$ in the risk-free asset has the following expected return and volatility:

$$\begin{aligned} \mu_P &= (1-w)r_f + w\mu_Q, \\ \sigma_P &= |w|\sigma_Q. \end{aligned} \tag{1}$$

If we only consider portfolios in which we invest in the risky asset, i.e. $w \geq 0$, we can combine both equations to get

$$\mu_P = r_f + \left(\frac{\mu_Q - r_f}{\sigma_Q}\right)\sigma_P.$$

The Sharpe ratio of the risky-asset is defined as:

$$SR = \frac{\mu_Q - r_f}{\sigma_Q}. \tag{2}$$

Therefore, if the x-axis is represented by σ and the y-axis is represented by μ, the expected return of any portfolio formed by combining the risk-free and the risky asset is given by a line with intercept r_f and slope coefficient *SR*:

$$\mu = r_f + SR \times \sigma.$$

This line is called the *Capital Allocation Line* of Q or just CAL(Q). When the risky-asset is the market portfolio the CAL(Q) is called the *Capital Market Line* (CML).

The CML transforms risk into expected returns. The CML can then be seen as a *production function* where the input is *risk* and the output is *expected return*. Thus, the Sharpe ratio of the market is the *marginal rate of transformation* (MRT) of risk into expected return.

Example 1. Suppose that $r_f = 5\%$, $\mu_Q = 12\%$ and $\sigma_Q = 20\%$. The Sharpe ratio of Q is

$$SR = \frac{0.12 - 0.05}{0.20} = 0.35.$$

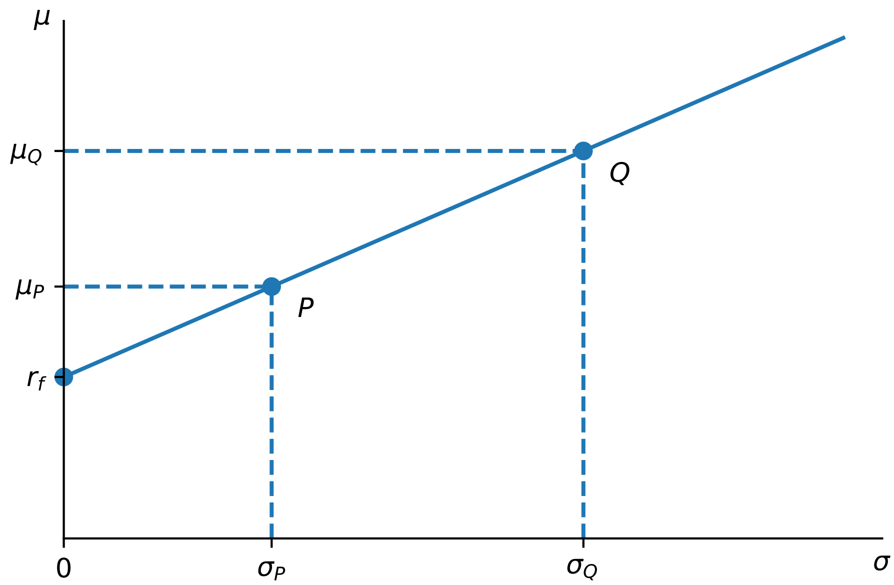

Figure 1: The figure shows the capital market line.

Suppose that you want a portfolio P on the CML but with 5% volatility. If w denotes the weight in the market, this means that

$$0.05 = w \times 0.20,$$

or $w = 25\%$. Thus, a portfolio that invest 25% in Q and 75% in the risk-free asset has 5% volatility. The expected return of this portfolio is:

$$\mu_P = 0.75 \times 0.05 + 0.25 \times 0.12 = 7.8\%.$$

The position of P in the CML is illustrated in Figure 1. □

Investor's Utility

Investors seek to get the maximum return for the minimum risk. A standard way in economics to capture this trade-off is by using a *utility function*. The idea of introducing a utility function is to be able to rank different combinations of risk vs. return. A simple way to do this in finance is to define:

$$U(\mu, \sigma) = \mu - \frac{1}{2}A\sigma^2.$$

The coefficient A denotes how sensitive is a particular investor to risk measured here by σ^2. A higher value for A reduces the utility for the same level of risk. We call A the coefficient of risk-aversion. It is common in applications to use values for A between 1 and 4.

This means that there are several pairs (μ, σ) that provide the same utility, i.e. for a given U and σ, we can always find a μ such that:

$$\mu = U + \frac{1}{2}A\sigma^2.$$

These functions are called *indifference curves* since an investor with risk-aversion coefficient A is indifferent among any of these combinations of μ and σ.

Let's fix $A = 3$ and U to be either 2%, 6% or 10%. We can now plot the corresponding indifference curves.

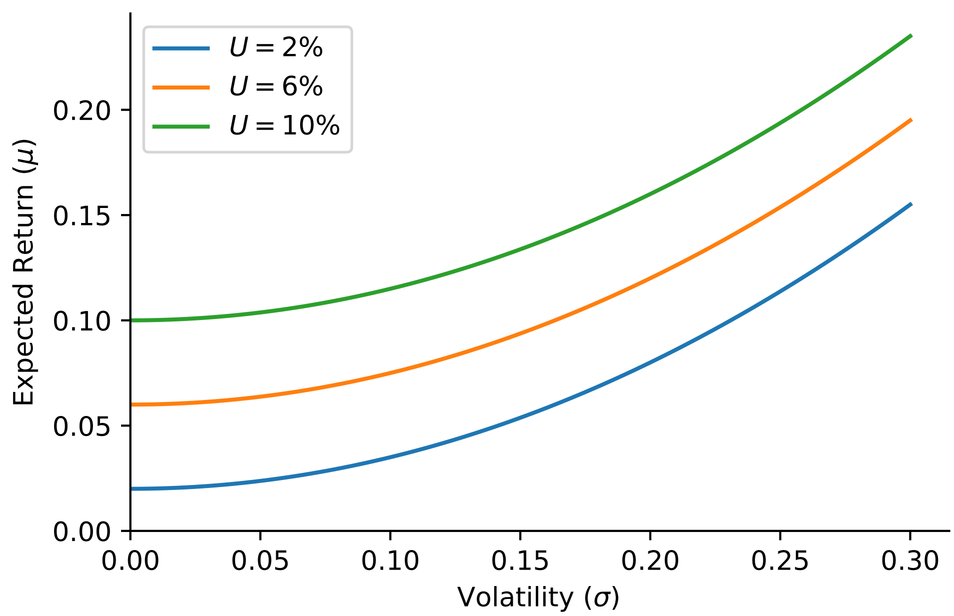

Figure 2: The figure shows indifferece curves for different levels of utility.

The curves in the graph represent all combinations of (μ, σ) that provide the same utility, i.e. the investor is *indifferent* among these choices of risk and return. Indifference curves that provide higher utility are always above indifference curves that provide lower utility.

Each indifference curve can be characterized by its *certainty equivalent*, which represents the expected return that would provide the same level of utility with no risk, that is when $\sigma = 0$. The utility level can therefore be interpreted as the certainty equivalent of a particular portfolio.

Maximizing Utility

Optimal portfolio choice is about maximizing utility given the constraints imposed by the investment opportunity set. For a given w that determines the weight in the risk asset Q, the investment opportunity set is characterized by:

$$\begin{aligned} \mu &= (1-w)r_F + w\mu_Q, \\ \sigma^2 &= w^2\sigma_Q^2. \end{aligned} \tag{3}$$

The utility of investing w in Q and the rest in the risk-free asset :

$$\begin{aligned} U &= \mu - \frac{1}{2}A\sigma^2 \\ &= (1-w)r_F + w\mu_Q - \frac{1}{2}Aw^2\sigma_Q^2. \end{aligned}$$

The first-order condition (FOC) is:

$$\frac{dU}{dw} = (\mu_Q - r_F) - Aw\sigma_Q^2 = 0.$$

implying that the optimal w^* is given by:

$$w^* = \frac{\mu_Q - r_F}{A\sigma_Q^2}.$$

The previous expression shows that the amount allocated to the risky asset is smaller if the risk aversion or if its variance are larger. We can see that in terms of allocation to the risky asset, the investor's risk aversion and the variance of the asset play the same role and are indistinguishable. Certainly, we can also see that the mount allocated to the risky asset increases with its expected return, but decreases with the risk-free rate.[1]

Knowing w^* tells us how much to invest in the risky asset and therefore how much to invest in the risk-free asset. The resulting expected return and standard deviation of the optimal portfolio are given by:

$$\mu^* = (1 - w^*)r_F + w^*\mu_Q,$$
$$\sigma^* = w^*\sigma_Q.$$

Example 2. Consider an agent with a risk-aversion coefficient equal to 3. If $r_f = 5\%$, $\mu_Q = 12\%$ and $\sigma_Q = 20\%$ as in Example 1, we have that

$$w^* = \frac{0.12 - 0.05}{3 \times 0.20^2} = 58.33\%.$$

Therefore, the portfolio that maximizes the utility for the investor consists investing 41.67% in the risk-free asset and 58.33% in the risky asset. The expected return and volatility of this portfolio are

$$\mu^* = (1 - w^*) \times 0.05 + w^* \times 0.12 = 9.08\%,$$
$$\sigma^* = w^* \times 0.20 = 11.67\%.$$

□

The optimal portfolio has the following interpretation. The investor wants to maximize utility, i.e., to achieve the highest level of utility possible. The point where the indifference curve is tangent to the capital market line determines the optimal portfolio. At this point, the *marginal rate of substitution* between risk and return equals the *marginal rate of transformation* between risk and return.

[1]We will see later that if Q is the market portfolio, in equilibrium we must have that $w^* = 1$. Therefore, an increase in σ_M^2 or A will translate in an increase of μ_M, i.e. a decrease in prices.

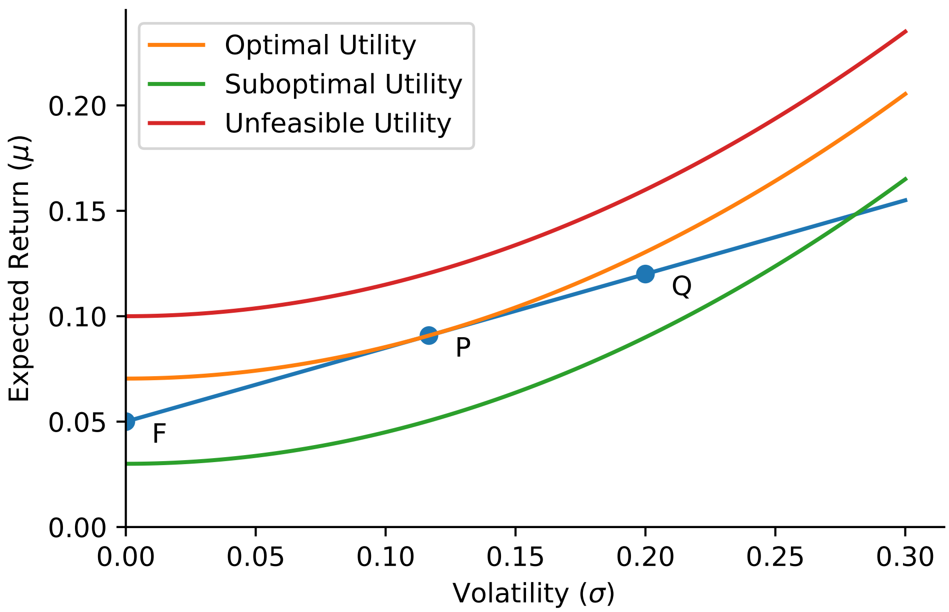

Figure 3: The figure shows that optimal portfolio choice occurs where the marginal rate of substituion equals the marginal rate of transformation between risk and return.