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The Treynor-Black Model

Model Setup

The model presented in this note is from Treynor and Black (1973). We have n risky assets with excess returns following a single-index model

$$R_i = \alpha_i + \beta_i R_m + e_i,$$

where $\mathrm{E}(e_i) = 0$, $\mathrm{Cov}(R_m, e_i) = 0$ and $\mathrm{Cov}(e_i, e_j) = 0$ for $i \neq j$, for all $i, j = 1, \ldots, n$. Suppose we form a portfolio P with the n risky assets and the market portfolio M.

Denote by w_i the weights of the n risky assets and w_M the weight in the market portfolio. Of course, the sum of the portfolio weights must be equal to one, i.e.,

$$\sum_{i=1}^{n} w_i + w_M = 1. \tag{1}$$

We can write the excess returns of this portfolio over the risk-free asset as

$$R_P = \alpha_P + \beta_P R_M + e_P.$$

The portfolio alpha includes the alpha of all the securities except the market that has no alpha. Therefore,

$$\alpha_P = \sum_{i=1}^{n} w_i \alpha_i. \tag{2}$$

We will see later that the alpha of the optimal portfolio is positive since its weights are positive for securities with positive alpha, and negative otherwise.

The beta of this portfolio is a weighted average of the beta of all securities and the beta of the market, which is one. Thus,

$$\beta_P = \sum_{i=1}^{n} w_i \beta_i + w_M. \tag{3}$$

For simplicity, in the following we will not replace immediately (3) into the expressions for the expected return and variance of the portfolio. It is apparent that choosing w_M given the weights of the other assets is the same as choosing β_P. Therefore, our optimization procedure will choose to maximize the Sharpe ratio of the target portfolio by choosing the weights in the individual assets and the beta of the target portfolio.

The idiosyncratic risk includes the firm-specific risks of all securities except the market that has no idiosyncratic risk. Because the firm-specific risks are uncorrelated with each other, we have that

$$\sigma^2(e_P) = \sum_{i=1}^{n} w_i^2 \sigma^2(e_i). \tag{4}$$

Thus,

$$\mathrm{E}(R_P) = \sum_{i=1}^{n} w_i \alpha_i + \beta_P \, \mathrm{E}(R_M), \tag{5}$$

and

$$\sigma_P^2 = \beta_P^2 \sigma_M^2 + \sum_{i=1}^{n} w_i^2 \sigma^2(e_i). \tag{6}$$

The expected return and variance in (5) and (6) depend on the alphas generated by the active securities and the systematic exposure of the resulting portfolio. This is the essence of the Treynor-Black model. The active part of portfolio, which might behave like a zero-cost portfolio, generates an alpha but might increases the total risk by adding diversifiable risk. It might also increase the expected return by loading on systematic risk, and therefore increase the systematic risk of the resulting portfolio.

Solving the Model

The objective is to find the portfolio weights $w_1, w_2, \dots, w_n$ and w_M that maximize its Sharpe ratio. Since $\beta_P = \sum_{i=1}^{n} w_i \beta_i + w_M$, this is equivalent to finding $w_1, w_2, \dots, w_n$ and the beta of the final portfolio that maximize the Sharpe ratio of the target portfolio.

Maximizing the Sharpe ratio is equivalent to minimizing the variance of the portfolio for a given expected return. Of course, only one particular expected return satisfies the restriction established in (1). For other targets of the expected return, we would have to borrow or invest the difference at the risk-free rate. If we denote by μ_P a possible target expected return, the minimization problem is as follows:

$$\begin{aligned} \min_{\{w_1, w_2, \dots, w_n, \beta_P\}} \quad & \frac{1}{2}\sigma_P^2 \\ \text{s.t.} \quad & \mathrm{E}(R_P) = \mu_P - r_f. \end{aligned}$$

The FOCs are

$$\begin{aligned} w_i \sigma^2(e_i) - \lambda \alpha_i &= 0, \forall i = 1, 2, \dots, n \\ \beta_P \sigma_M^2 - \lambda\, \mathrm{E}(R_M) &= 0, \\ \mathrm{E}(R_P) - (\mu_P - r_f) &= 0. \end{aligned}$$

The last FOC is not important since we will compute λ later by imposing the condition that the sum of the weights is equal to one. The first FOC implies that for each security we have that

$$w_i = \lambda \frac{\alpha_i}{\sigma^2(e_i)}. \tag{7}$$

The expression says that the weight of each security in the active portfolio increases with the alpha but decreases with the variance of the firm-specific risk. In other words, the weight of each security in the active portfolio depends on the *quality* of the alpha. An small alpha with little firm-specific variance can be a great arbitrage compared to a large alpha with a large variance of the idiosyncratic risk. Assets with negative alphas carry a negative portfolio weight whereas assets with positive alphas have a positive weight in the final portfolio.

Using the weights of the individual assets, the weight in the market portfolio can be computed as

$$w_M = 1 - \lambda \sum_{i=1}^{n} \frac{\alpha_i}{\sigma^2(e_i)}. \tag{8}$$

The second FOC implies that

$$\beta_P = \lambda \frac{\mathrm{E}\, R_M}{\sigma_M^2}. \tag{9}$$

On the other hand, from the definition of the portfolio beta we get that

$$\beta_P = \lambda \sum_{i=1}^{n} \frac{\alpha_i}{\sigma^2(e_i)} \beta_i + w_M.$$

Using (8) in the previous expression we get

$$\beta_P = \lambda \sum_{i=1}^{n} \frac{\alpha_i}{\sigma^2(e_i)} (\beta_i - 1) + 1. \tag{10}$$

Combining (9) with (10) we get

$$\lambda = \frac{1}{\frac{\mathrm{E}\, R_M}{\sigma_M^2} + \sum_{i=1}^{n} \frac{\alpha_i}{\sigma^2(e_i)} (1 - \beta_i)}. \tag{11}$$

The previous equation provides an expression for λ that can be computed using the model inputs. The value for λ can then be used to compute the weights of the individual assets and of the market portfolio.

The Information Ratio

We can use (7) and (9) in (5) to get an expression of the expected return of the optimal portfolio:

$$\mathrm{E}(R_P) = \lambda \left[\sum_{i=1}^{n} \left(\frac{\alpha_i}{\sigma(e_i)} \right)^2 + \left(\frac{\mathrm{E}(R_M)}{\sigma_M} \right)^2 \right]. \tag{12}$$

Similarly, (6) allows us to compute the optimal portfolio variance, i.e.,

$$\sigma_P^2 = \lambda^2 \left[\sum_{i=1}^{n} \left(\frac{\alpha_i}{\sigma(e_i)} \right)^2 + \left(\frac{\mathsf{E}(R_M)}{\sigma_M} \right)^2 \right]. \tag{13}$$

These two expressions above yield the following relationship between the Sharpe ratio of the optimal portfolio to the market Sharpe ratio,

$$\left(\frac{\mathsf{E}(R_P)}{\sigma_P} \right)^2 = \sum_{i=1}^{n} \left(\frac{\alpha_i}{\sigma(e_i)} \right)^2 + \left(\frac{\mathsf{E}(R_M)}{\sigma_M} \right)^2. \tag{14}$$

The equation shows that the Sharpe ratio of the optimal portfolio is higher than the Sharpe ratio of the market alone. Each security contributes positively in increasing the Sharpe ratio of the optimal portfolio, regardless of whether the alpha is positive or negative. Remember that securities with negative alphas have negative weights, i.e., they are sold short in order to take advantage of the fact that they are expensive relative to the model.

For a given security i, the term $\alpha_i/\sigma(e_i)$ is called its **information ratio**, and measures how *good* the alpha is. If the volatility of the firm-specific risk is small, then it will be easier to diversify that risk away and hence extract the alpha of the security. Each security contributes to increasing the squared value of the Sharpe ratio of the optimal portfolio by the square of its information ratio.

References

Treynor, Jack L., and Fischer Black. 1973. "How to Use Security Analysis to Improve Portfolio Selection." *Journal of Business* 46 (1): 66–86.