Statistics of Asset Returns

The Rate of Return

Investing involves purchasing an asset today and selling it next period. The asset can be a real asset, like a house, or a financial asset like a stock or a bond. All assets in the economy are real assets. Financial assets are claims to parts of real assets.

When you purchase an asset today and hold it for one period, you might also be entitled to a dividend. Many stocks in the U.S. pay dividends quarterly, while bonds pay coupons semi-annually. Even real assets can pay a dividend. For example, if you purchase a house and rent it, you will collect a monthly rent from your tenant.

In order to analyze the performance of your investment, it is useful to think in terms of profitability per dollar invested. Say you purchase an asset at time 0 for P_0 and sell it a period later for P_1. If the stock pays dividends, we assume that dividends are paid just before you purchase or sell the asset. The holding-period return (HPR) from time 0 to time 1 is defined as

$$\text{HPR} = \frac{P_1 - P_0 + D_1}{P_0}.$$

The HPR can be decomposed as

$$\text{HPR} = \frac{P_1 - P_0}{P_0} + \frac{D_1}{P_0}.$$

The first term in the previous expression denotes the *capital gain* and the second term represents the dividend yield. Thus, the total return of investing in the asset depends on how much the price appreciates or depreciates, and the amount of dividends paid during the holding period.

Example 1. Suppose you purchase today an index fund for \$100 per share. The fund will pay dividends over a year of \$4. If the price per share next year is \$110, your HPR amounts to

$$\text{HPR} = \frac{110 - 100 + 4}{100} = 14\%.$$

The total return of 14% consists of 10% in capital gains and 4% of dividend yield. □

In order to ease notation, in the following we denote by r_i the HPR of investing in security i. Since there is considerable uncertainty about the future price and dividends paid by a stock, its HPR is unknown today. A useful way to think about the uncertainty of r_i is to model it as a random variable in certain probability space (Ω, P).

The expectation of r_i is called the expected return of security i, whereas the standard deviation of r_i is typically called the volatility or standard deviation of returns. I usually write

$$\mu_i = \mathrm{E}(r_i),$$
$$\sigma_i = \sqrt{\mathrm{V}(r_i)}.$$

Example 2. Suppose your expectations regarding a stock price are as follows:

State of the Market	Probability	HPR
Boom	0.35	44.5%
Normal	0.30	14%
Recession	0.35	-16.5%

We can compute the mean and standard deviation of the stock's HPR as

$$\begin{aligned}\mu &= 0.35 \times 0.445 + 0.3 \times 0.14 + 0.35 \times (-0.165) = 14\%,\\ \sigma &= \sqrt{0.35 \times (0.445 - 0.14)^2 + 0.3 \times (0.14 - 0.14)^2 + 0.35 \times (-0.165 - 0.14)^2}\\ &= 25.52\%.\end{aligned}$$

□

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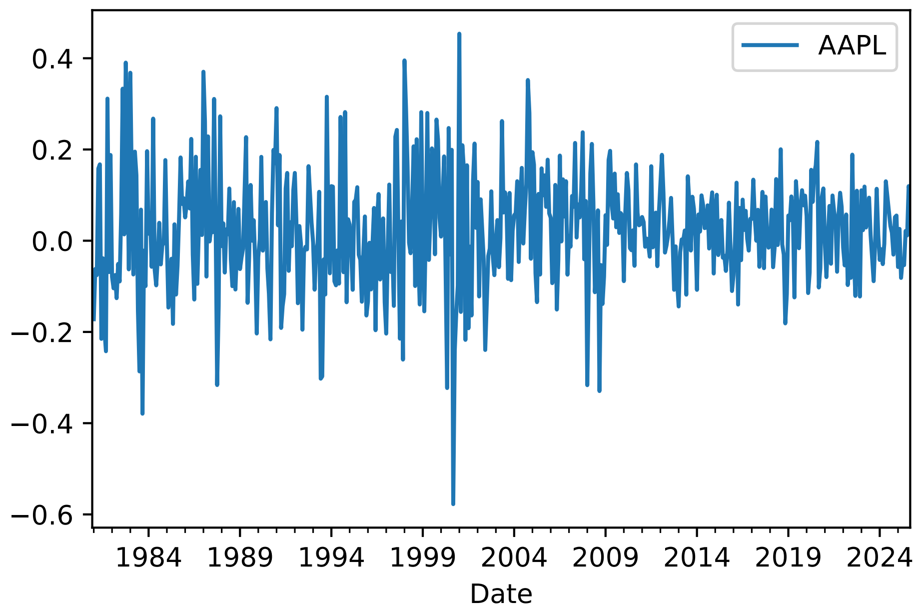

Figure 1: The figure shows the monthly returns of Apple (AAPL) since the stock was listed in December 1980.

Figure 1 plots the monthly returns of Apple (AAPL) since the stock was listed in December 1980. The figure shows that returns are volatile with periods in which volatility spikes and other periods of lower volatility. We can compute the average monthly return as

$$\bar{r} = \frac{1}{N}\sum_{t=1}^{N} r_t,$$

and sample standard deviation as

$$\hat{\sigma} = \sqrt{\frac{1}{N-1}\sum_{t=1}^{N}(r_t - \bar{r})}.$$

If we assume that the monthly returns are independent of each other, we can annualize the average monthly return and volatility as:

$$\bar{r}_{\text{Annual}} = 12 \times \bar{r}_{\text{Monthly}},$$
$$\hat{\sigma}_{\text{Annual}} = \sqrt{12} \times \hat{\sigma}_{\text{Monthly}}.$$

Using these expressions, we get for Apple that

	Monthly Estimate (%)	Annualized Estimate (%)
Mean	2.25	26.97
St. Dev.	12.59	43.61

Because the volatility is so high compared to the average return, estimates of average returns are typically very noisy and not very useful in practice to proxy for expected returns. Also, an average of monthly returns going back to 1980 is probably not a very good proxy of your *expectation* for returns going forward.

Portfolios

In finance, a *portfolio* consists of allocating a certain amount of wealth into different assets such as stocks, bonds, and real estate. Say you have two assets A and B. Denote by r_A and r_B the HPR of each asset, respectively. If you have a certain amount of wealth W, you can split it and invest W_A in asset A and the rest, $W_B = W - W_A$, in B. Each dollar invested in A yields $1 + r_A$ dollars next period whereas each dollar invested in B generates $1 + r_B$.

Denote by r the HPR of your portfolio. Using this notation, the total return of your investment next period is $(1 + r)W$. But this amount can be computed by considering the returns of the individual investments in A and B. Therefore, we must have that

$$(1 + r)W = (1 + r_A)W_A + (1 + r_B)W_B.$$

Since $W = W_A + W_B$, the previous expression can be simplified as

$$r = \frac{W_A}{W} r_A + \frac{W_B}{W} r_B. \tag{1}$$

In equation (1), the fractions W_A/W and W_B/W denote the proportion of wealth allocated to each asset. In finance, we call these fractions the portfolio weights. If we denote by w_A and w_B these portfolio weights, we can write expression (1) as

$$r = w_A r_A + w_B r_B.$$

An important thing to note is that the sum of the weights is by construction always equal to one.

Example 3. You have the following scenario analysis for the HPR of stocks X and Y:

Market	Probability	Stock X	Stock Y
Bull	0.3	40%	10%
Normal	0.5	15%	20%
Bear	0.2	-18%	-5%

Assume that of your $10,000 portfolio, you invest $8,000 in Stock X and $2,000 in Stock Y.

To compute the expected return and standard deviation of the portfolio, we can first compute the HPR in each state of the world:

Market	Probability	Invested in X	Invested in Y	Total	HPR
Bull	0.3	11,200	2,200	13,400	34.0%
Normal	0.5	9,200	2,400	11,600	16.0%
Bear	0.2	6,560	1,900	8,460	-15.4%

Thus,

$$\mu_P = 0.3 \times 0.34 + 0.5 \times 0.16 + 0.2 \times (-0.154) = 15.12\%,$$
$$\sigma_P = \sqrt{0.3 \times (0.34 - 0.1512)^2 + 0.5 \times (0.16 - 0.1512)^2 + 0.2 \times (-0.154 - 0.1512)^2}$$
$$= 17.14\%.$$

□

The portfolio mathematics we just did relies on a very important assumption called the **law of one price** (LOOP). In competitive markets, LOOP guarantees that the price of a basket of stocks is equal to the sum of the prices of its constituents. This logic is at the heart of how **Exchange-Traded Funds** (ETF) operate, as the next example shows.

Example 4. An ETF is a type of investment fund that is traded on stock exchanges, similar to individual stocks. ETFs hold a diversified portfolio of assets, such as stocks, bonds, or commodities, which provides investors with broad exposure to specific markets or investment strategies.

ETF arbitrage is the mechanism that helps keep the market price of an ETF in line with its Net Asset Value (NAV). Authorized Participants (APs), typically large financial institutions, have the ability to create or redeem ETF shares in large blocks called creation units.

When the ETF market price is higher than the NAV, APs can buy the underlying securities of the ETF in the open market and then deliver them to the ETF issuer in exchange for new ETF shares. The AP can then sell these ETF shares at the higher market price, making a profit. This buying of underlying securities pushes their prices up, while the selling of new ETF shares pushes the ETF price down, bringing the two prices closer together.

When the ETF market price is lower than the NAV, APs can buy ETF shares in the open market and deliver them to the ETF issuer in exchange for the underlying securities. The AP can then sell these underlying securities at the higher NAV price, making a profit. This buying of ETF shares pushes their price up, while the selling of the underlying securities pushes their prices down, again bringing the two prices closer together.

This creation and redemption process happens continuously and helps to keep the ETF price in line with the NAV. The arbitrage opportunities are typically small but sufficient for APs to engage in the process for profit, ensuring that the ETF price does not deviate significantly from its NAV.

You can find more information here. □

The fact that the market for ETFs is so liquid and works flawlessly reassures us that LOOP is a reasonable axiom to start working from.

One of the most important ETFs out there is the SPDR S&P 500 ETF Trust which seeks to provide investment results that, before expenses, correspond generally to the price and yield performance of the S&P 500 Index. The figure below plots the monthly returns of SPDR (Ticker: SPY) since the ETF was listed in 1993.

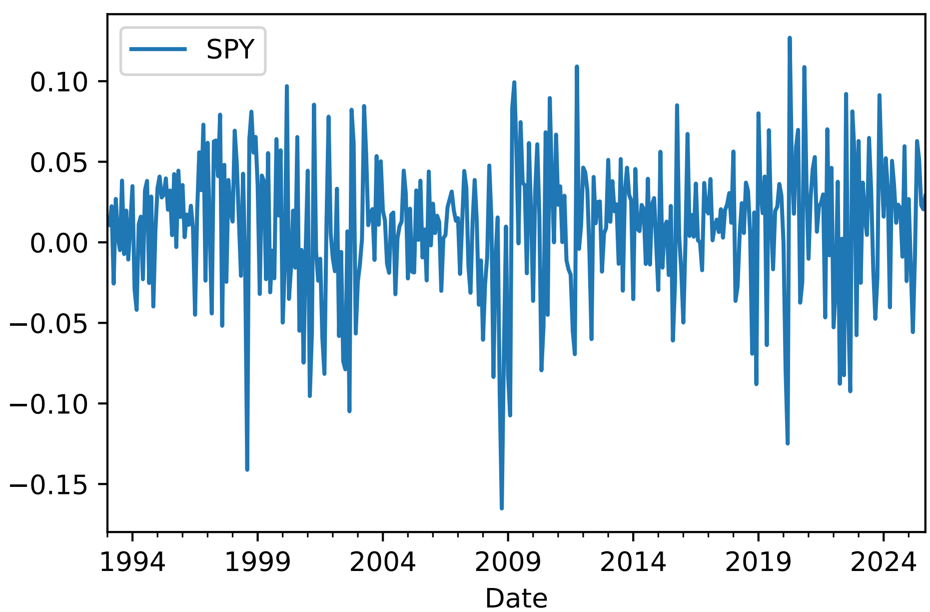

Figure 2: The figure shows the monthly returns of SPDR (SPY) since the ETF was listed in 1993.