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Risky Portfolios and the CAPM

A Portfolio of Two Risky Assets

Suppose we have two risky assets A and B. The return of a portfolio P in which we invest $1 - w$ in A and w in B is

$$r_P = (1 - w)r_A + wr_B. \tag{1}$$

The expected return of the portfolio is

$$\mu_P = (1 - w)\mu_A + w\mu_B, \tag{2}$$

whereas its variance can be computed as

$$\begin{aligned} \sigma_P^2 &= \mathsf{V}((1 - w)r_A + wr_B) \\ &= (1 - w)^2\sigma_A^2 + w^2\sigma_B^2 + 2w(1 - w)\sigma_{A,B}. \end{aligned} \tag{3}$$

Here $\sigma_{A,B}$ denotes the covariance of returns between A and B.

Example 1 (A Portfolio with Two Risky Assets)**.** Consider two risky assets A and B for which you have the following information.

Asset	Expected Return	Standard Deviation
A	10%	20%
B	15%	35%

The correlation between the assets returns is 0.4. If you invest 40% in A and 60% in B, your portfolio will have an expected return of

$$\mu = 0.4 \times 0.10 + 0.6 \times 0.15 = 13.0\%.$$

Since the covariance of returns is $\sigma_{AB} = 0.20 \times 0.35 \times 0.4 = 0.028$, the standard deviation of the portfolio returns is

$$\sigma = \sqrt{0.4^2 \times 0.20^2 + 0.6^2 \times 0.35^2 + 2 \times 0.4 \times 0.6 \times 0.028} = 25.29\%.$$

□

Suppose that now we want to vary w and see all the possible combinations of expected return (μ_P) and standard deviation (σ_P) that we can obtain. In the previous example, we split the wealth into the two assets so that $0 \leq w \leq 1$ and implies that $0 \leq 1 - w \leq 1$. What if we wanted to invest more than 100% in asset B?

In financial markets, it is possible to borrow an asset and then sell it. We call this transaction short-selling the asset. Typically, there is no specific date when the asset must be paid back, but must be paid back as soon as the lender requires it.

In this class, we assume that short-selling is allowed. Short selling implies that w can be greater than one, in which case we borrow A to overinvest in B, or less than zero, in which case we borrow B to overinvest in A.

By varying w from $-\infty$ to ∞, we obtain what is called the **investment opportunity set** generated by the two risky assets. If we plot μ_P in the y-axis as a function of σ_P in the x-axis, the resulting figure is an hyperbola.

The Minimum Variance Portfolio (MVP)

Definition

As shown in the previous plot, there is a portfolio that has the minimum variance among all portfolios between A and B. To find its composition, we can use standard optimization techniques:

$$\begin{aligned} \frac{d}{dw}\sigma_P^2 &= \frac{d}{dw}(1-w)^2\sigma_A^2 + w^2\sigma_B^2 + 2w(1-w)\sigma_{A,B} \\ &= -2(1-w)\sigma_A^2 + 2w\sigma_B^2 + 2(1-2w)\sigma_{A,B} = 0. \end{aligned} \tag{4}$$

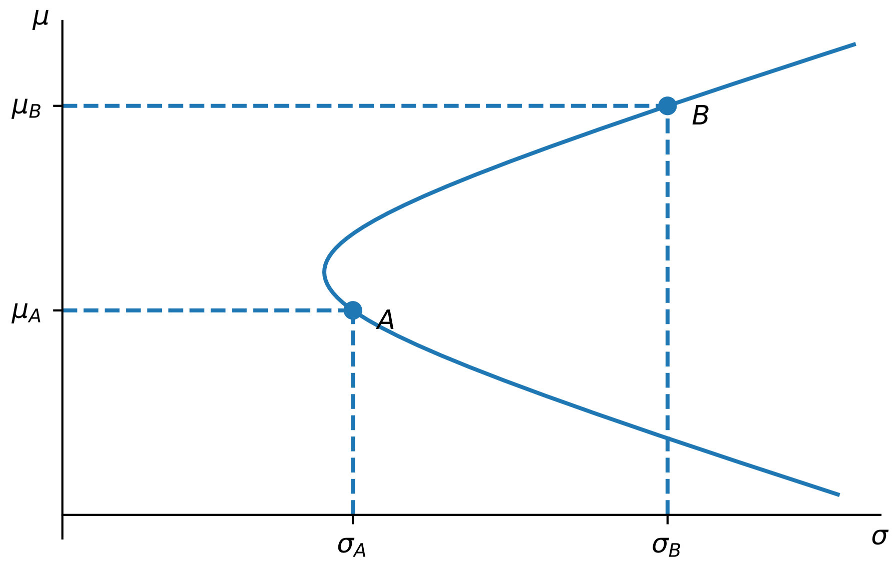

Figure 1: The figure shows the investment opportunity set generated by two risky assets A and B.

Thus,

$$-(1-w)\sigma_A^2 + w\sigma_B^2 + (1-2w)\sigma_{A,B} = 0.$$

or

$$w_{MV} = \frac{\sigma_A^2 - \sigma_{A,B}}{\sigma_A^2 + \sigma_B^2 - 2\sigma_{A,B}}. \tag{5}$$

Property 1 (The Minimum Variance Portfolio)**.** *Given two risky assets A and B, there is a portfolio that has the minimum variance among all possible portfolios that can be built with A and B. The weights of the minimum variance portfolio are given by*

$$w_A = \frac{\sigma_B^2 - \sigma_{AB}}{\sigma_A^2 + \sigma_B^2 - 2\sigma_{AB}},$$
$$w_B = \frac{\sigma_A^2 - \sigma_{AB}}{\sigma_A^2 + \sigma_B^2 - 2\sigma_{AB}}.$$

In the previous expression, σ_A and σ_B denote the standard deviation or volatility of returns. The term σ_{AB} denotes the covariance of A and B and is equal to

$$\sigma_{AB} = \sigma_A\sigma_B\rho_{AB},$$

where ρ_{AB} is the correlation between the returns of A and B.

Example 2 (Minimum Variance Portfolio)**.** Using the data of Example 1, we find that the weights of the minimum variance portfolio are given by

$$w_A = \frac{0.35^2 - 0.028}{0.20^2 + 0.35^2 - 2 \times 0.028} = 88.73\%$$
$$w_B = \frac{0.20^2 - 0.028}{0.20^2 + 0.35^2 - 2 \times 0.028} = 11.27\%$$

Thus, the expected return and volatility of the the minimum variance portfolio are

$$\begin{aligned}\mu &= 0.8873 \times 0.10 + 0.1127 \times 0.15 = 10.56\%,\\ \sigma &= \sqrt{0.8873^2 \times 0.20^2 + 0.1127^2 \times 0.35^2 + 2 \times 0.8873 \times 0.1127 \times 0.028}\\ &= 19.66\%.\end{aligned}$$

□

The MVP is the point farthest to the left of the investment opportunity set.

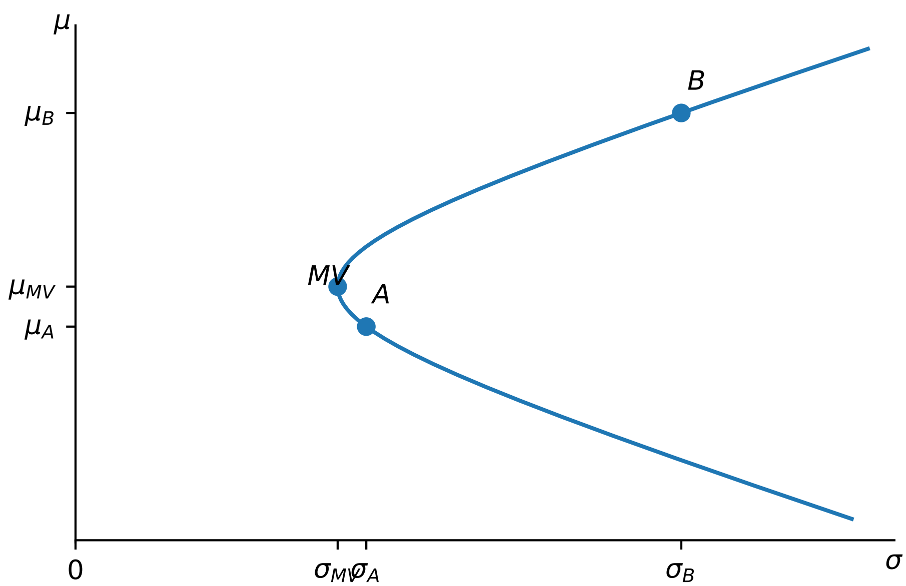

Figure 2: The figure shows the minimum variance portfolio obtained from combining two risky assets.

Properties of the Minimum Variance Portfolio

By definition, any other portfolio P on the investment set has a variance greater than the MVP, i.e. $\sigma_P^2 \geq \sigma_{MV}^2$. Thus, we can write σ_P^2 as σ_{MV}^2 plus something positive. That additional positive term must be the variance of some portfolio Z. We can then write

$$\sigma_P^2 = \sigma_{MV}^2 + \sigma_Z^2.$$

The previous expression implies that

$$r_P = r_{MV} + r_Z$$

where $r_Z = r_P - r_{MV}$ is a zero-cost portfolio orthogonal to r_{MV}, i.e.,

$$\text{Cov}(r_Z, r_{MV}) = 0.$$

Thus,

$$\text{Cov}(r_P, r_{MV}) = \sigma_{MV}^2.$$

The covariance of the MVP with any other portfolio is always the same and equal to its own variance! This surprising result also implies that the correlation of the MVP with any other portfolio is always positive.

Example 3 (Computing a Correlation with the MVP)**.** Consider an investment opportunity set where the MVP has a standard deviation of returns of 20%. An asset A has a standard deviation of returns equal to 30%. Thus,

$$\sigma_A \sigma_{MV} \rho_{A,MV} = \sigma_{A,MV} = \sigma_{MV}^2,$$

or

$$\rho_{A,MV} = \frac{\sigma_{MV}}{\sigma_A} = \frac{2}{3}.$$

□

Equation (5) demonstrates that the weights of the MVP depend solely on variances and covariances, which can typically be estimated with high precision from the data. In contrast, constructing any other portfolio within the investment opportunity set requires estimating expected returns, which are generally more challenging to estimate accurately. This is the reason why some practitioners like to use the MVP for their asset allocation. Moreover, imposing short-sale constraints to the estimation of the minimum-variance portfolio makes the model more robust. Jagannathan and Ma (2003) show that with no short-sale constraints in place, the sample covariance matrix performs as well as covariance matrix estimates based on factor models, shrinkage estimators, and daily data.

An Equally-Weighted Portfolio

The minimum-variance frontier demonstrates that combining two risky assets can alter the risk-return profile of the original assets. This concept is known as **portfolio diversification**.

To understand this phenomenon better, say that we have N securities and we invest the same amount in each, so the return of this portfolio (that we call P) is

$$r_P = \frac{1}{N}\sum_{i=1}^{N} r_i.$$

The variance of P is given by

$$\sigma_P^2 = \frac{1}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N} \text{Cov}(r_i, r_j). \tag{6}$$

The previous expression looks complex, but it is possible to decompose it into two intuitive parts. First, note that the average variance of the securities is given by

$$\text{Avg. Variance} = \frac{1}{N}\sum_{i=1}^{N} \text{V}(r_i).$$

Furthermore, the average covariance of the securities can be computed as

$$\text{Avg. Covariance} = \frac{1}{N(N-1)} \sum_{i=1}^{N} \sum_{\substack{j=1 \\ i \neq j}}^{N} \text{Cov}(r_i, r_j)$$

Using these two quantities, we can write (6) as

$$\sigma_P^2 = \frac{1}{N}(\text{Average Variance}) + \frac{N-1}{N}(\text{Average Covariance}). \tag{7}$$

As N increases in (7), we have that

$$\sigma_P^2 \xrightarrow[N \to \infty]{} \text{Average Covariance}.$$

We can see that the average variance of the portfolio disappears! In other words, this is the variance that can be diversified away.

While diversification reduces risk, it is constrained by the degree to which assets move together. The average covariance between asset pairs sets a limit on the extent of risk reduction achievable through diversification.

This limitation became evident during the 2008 financial crisis involving subprime mortgages. Many financial institutions underestimated the covariance structure in default risks, leading to widespread failures. The minimum variance portfolio, while optimal in minimizing variance, cannot eliminate all risk due to these underlying correlations.

Adding a Risk-Free Asset

Adding a risk-free asset allows us to generate a CAL for each risky portfolio in our original investment opportunity set. Each of these CALs will determine the investment opportunity set generated by the risk-free asset and the two risky assets.

By moving the CALs higher and higher, we reach a limit on how high the Sharpe ratio of the CALs can be. Indeed, there is one portfolio Q that achieves the highest Sharpe

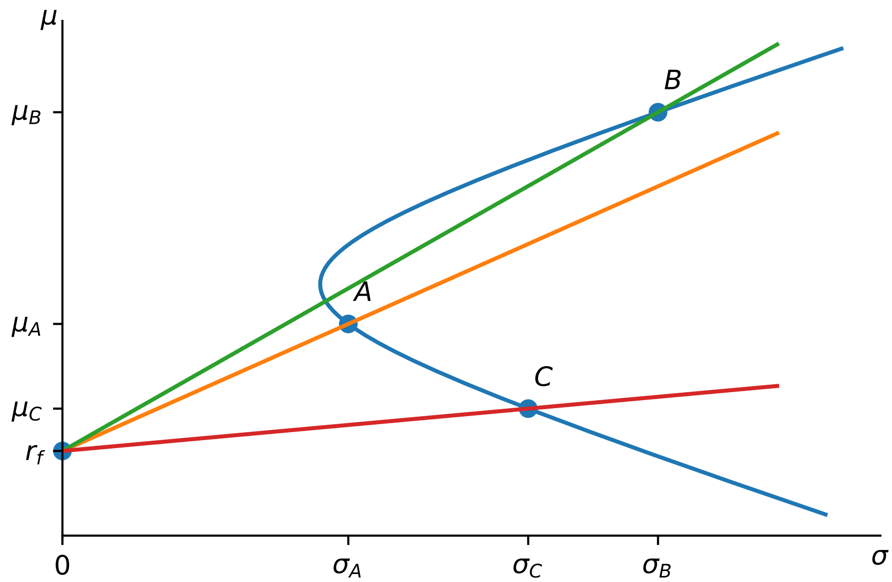

Figure 3: The figure shows the capital allocation lines of portfolios A, B, and C that are members of the same investment opportunity set of risky assets.

ratio. We call this portfolio the **tangency portfolio** because effectively is the portfolio that generates a CAL that is tangent to the original investment opportunity set generated by the two risky assets.

Investors that prefer portfolios with the highest Sharpe ratio will choose a portfolio located in the CAL of the tangency portfolio. We call this CAL the **efficient frontier**. All portfolios in this CAL are efficient since they all have the highest Sharpe ratio. Any two portfolios in this line are enough to generate all the other efficient portfolios. Any other portfolios in this economy will have lower Sharpe ratios than the tangency portfolio. To determine if a portfolio is efficient or not, just compute its Sharpe ratio and compare it to the Sharpe ratio of an efficient portfolio.

Example 4 (Testing for Efficiency)**.** You know that the tangency portfolio has an expected return of 15% with a standard deviation of returns of 20%. The risk-free rate is 5% per year. You have the following information of two risky assets.

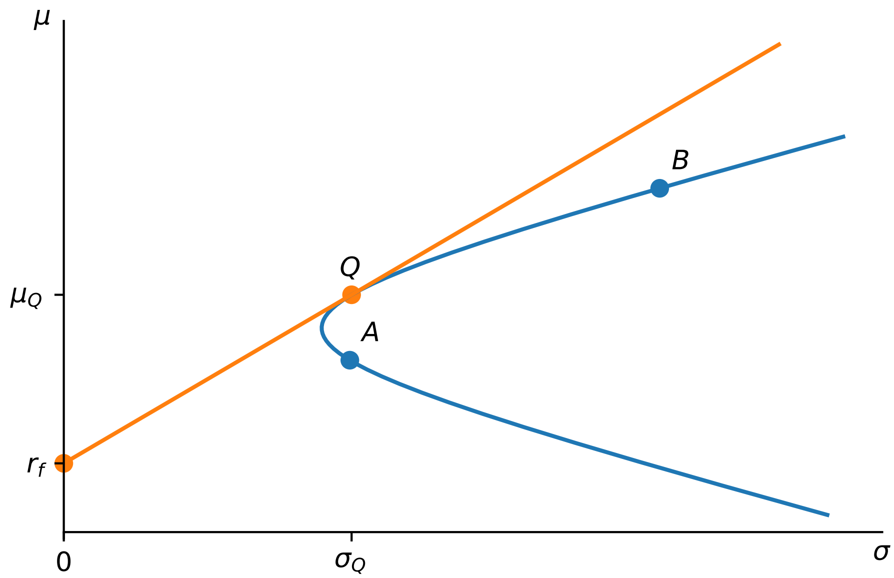

Figure 4: The figure shows the capital allocation line of the tangency portfolio.

Asset	Expected Return	Standard Deviation
A	10%	20%
B	25%	40%

Are these portfolios efficient? The Sharpe ratio of the tangency portfolio is $(15-5)/20 = 0.5$. The Sharpe ratio of A is $(10-5)/20 = 0.25 < 0.5$, whereas the Sharpe ratio of B is $(25-5)/40 = 0.5$. Thus only B is an efficient portfolio. □

Example 5 (Constructing an Efficient Portfolio). In Example 4, note that by investing 50% in the risk-free asset and 50% in the tangency portfolio, we obtain a portfolio with the same expected return as A but with a lower standard deviation of $0.5 \times 0.2 = 10\%$.

An investor who wants to achieve a 10% expected return would prefer to invest in this efficient portfolio rather than A. Less risk is always better! □

The Capital Asset Pricing Model

The previous analysis demonstrates that investors with utility functions of the form

$$U = \mu - \frac{1}{2}A\sigma^2$$

will allocate their investments between the tangency portfolio and the risk-free asset.

Since aggregate borrowing equals aggregate lending, all investors collectively hold the tangency portfolio in identical proportions, differing only in the scale of their investments. Consequently, the tangency portfolio represents the **market portfolio**, implying that the market portfolio is an efficient portfolio.

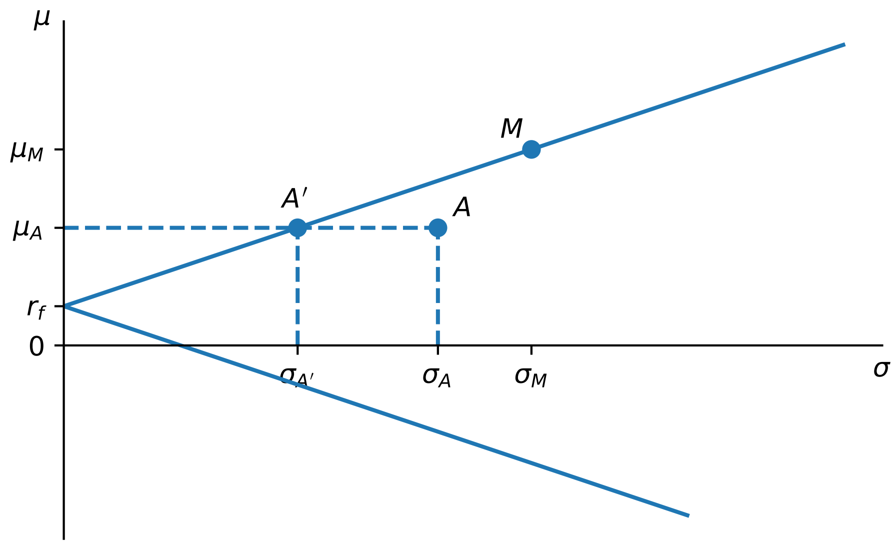

Figure 5: The figure shows the investment opportunity set available to an investor. The upper line determines the efficient frontier whereas the lower line is the set of inefficient portfolios. For a given portfolio A, there is a portfolio A' that has the same expected return as A but the lowest standard deviation. Both A' and M are efficient portfolios.

In Figure 5, portfolio A' is efficient, representing a combination of the market portfolio and the risk-free asset. Its return can be expressed as:

$$r_{A'} = (1 - \beta)r_f + \beta r_M.$$

The residual $\varepsilon = r_A - r_{A'}$ is constructed to have a mean of zero, leading to:

$$\mathsf{E}(r_A) = (1-\beta)r_f + \beta\,\mathsf{E}(r_M).$$

Additionally, the residual is orthogonal to its projection $r_{A'}$, which implies:

$$0 = \mathsf{Cov}(\varepsilon, r_{A'}) = \mathsf{Cov}(r_A - r_{A'}, r_{A'}) = \beta\,\mathsf{Cov}(r_A, r_M) - \beta^2\,\mathsf{V}(r_M),$$

or equivalently:

$$\beta = \frac{\mathsf{Cov}(r_A, r_M)}{\mathsf{V}(r_M)}.$$

This formula is commonly used to estimate the beta of an asset. By regressing the returns of the asset r_A on the returns of the market r_M, the slope coefficient obtained corresponds to the ratio of the covariance of r_A with r_M to the variance of r_M.

The CAPM further allows us to decompose returns into systematic and firm-specific components:

$$r_A = (1-\beta)r_f + \beta_A r_M + \varepsilon_A.$$

Here, the residual ε_A represents the idiosyncratic or firm-specific risk, which is diversifiable and orthogonal to market risk. The term $\beta_A r_M$ captures the systematic, non-diversifiable risk of the asset. A higher β indicates greater exposure to market risk.

If the CAPM does not hold, the analysis remains valid by replacing M with the tangency portfolio Q.

Property 2 (The Capital Asset Pricing Model)**.** *The returns of any asset can be decomposed into a systematic component which characterizes the non-diversifiable risk, and a firm-specific or idiosyncratic component containing the risk that can be diversified. Thus,*

$$r_A = (1-\beta)r_f + \beta_A r_M + \varepsilon_A, \tag{8}$$

where

$$\beta = \frac{\mathsf{Cov}(r_A, r_M)}{\mathsf{V}(r_M)}.$$

Since the risk in ε_A is not priced, the expected return of the asset depends on how the asset returns covary with the market risk,

$$\mathsf{E}(r_A) = (1-\beta)r_f + \beta\,\mathsf{E}(r_M). \tag{9}$$

Thus, according to the CAPM the only thing that determines the expected return of any risky asset is its covariance with the market portfolio, i.e. its beta.

Example 6 (Computing an Expected Return)**.** If the risk-free rate is 5%, $\beta_{DELL} = 1.3$, and $\mathsf{E}(r_M) = 14\%$, then the CAPM predicts that:

$$\mathsf{E}(r_{DELL}) = 0.05 + 1.3 \times (0.14 - 0.05) = 16.7\%.$$

Dell stock must have an expected annual return of 16.7%.

Note that the CAPM is a *prediction*, and tells us how much the price of the stock should increase on average next year. □

Example 7 (Computing a Stock Beta)**.** Suppose that you know that the correlation between stock A and the market is 0.6. If the standard deviation of A returns is 40% per year, and the standard deviation of the market is 20% per year, the beta of A is

$$\begin{aligned}\beta_A &= \frac{\mathsf{Cov}(r_A, r_M)}{\mathsf{V}(r_M)} = \frac{\sigma_A \sigma_M \rho_{A,M}}{\sigma_M^2} \\ &= \frac{\sigma_A \rho_{A,M}}{\sigma_M} = \frac{0.4 \times 0.6}{0.2} = 1.2.\end{aligned}$$

□

What happens if a security's expected return deviates from the CAPM prediction? One explanation could be market inefficiency, where the stock is mispriced, allowing some investors to exploit this anomaly while the broader market remains indifferent. Alternatively, investors might consider factors beyond mean and variance when selecting portfolios, such as other attributes of returns. In this scenario, the market portfolio would not be efficient, and the CAPM framework would not hold.

Ultimately, distinguishing between these two hypotheses using data alone is impossible. This highlights the dual hypothesis testing problem inherent in finance.

Finally, the return decomposition in equation (8) enables us to break down the variance of any asset into two components:

$$\sigma_A^2 = \beta_A^2 \sigma_M^2 + \sigma^2(\varepsilon_A). \tag{10}$$

Here, σ_A^2 represents the total variance of the asset, which can be divided into systematic variance, $\beta_A^2 \sigma_M^2$, and firm-specific variance, $\sigma^2(\varepsilon_A)$.

Example 8 (Computing the Residual Risk)**.** Suppose that stock A has a standard deviation of returns of 40% per year and a beta of 1.1 with the market. The standard deviation of the market returns is 20%.

This means that the residual variance is

$$\sigma^2(\varepsilon_A) = \sigma_A^2 - \beta_{A,M}^2 \sigma_M^2 = 0.1116.$$

Therefore, the standard deviation of the firm-specific risk is $\sqrt{0.1116} = 33.41\%$. □

References

Jagannathan, Ravi, and Tongshu Ma. 2003. "Risk Reduction in Large Portfolios: Why Imposing the Wrong Constraints Helps." *Journal of Finance* 58 (4): 1651–83.