Perfect Correlation

The Zero Variance Portfolio

When the correlation between two risky assets A and B is either one or minus one, we say that the assets are *perfectly correlated*. In this case, it is possible to create a portfolio P between the two assets with zero variance.

We saw in the previous note that the variance of a portfolio in which we invest w_A in A and w_B in B is

$$\sigma_P^2 = w_A^2\sigma_A^2 + w_B^2\sigma_B^2 + 2w_A w_B \sigma_{A,B}. \tag{1}$$

We start first considering the case where $\rho_{A,B} = 1$. In this case, we have that $\sigma_{A,B} = \sigma_A\sigma_B$ and we can write equation (1) as

$$\begin{aligned} \sigma_P^2 &= w_A^2\sigma_A^2 + w_B^2\sigma_B^2 + 2w_A w_B \sigma_A\sigma_B \\ &= (w_A\sigma_A + w_B\sigma_B)^2. \end{aligned} \tag{2}$$

Since $w_A = 1 - w_B$, equation (2) can be made equal to zero by using

$$w_B = \frac{\sigma_A}{\sigma_A - \sigma_B}.$$

Similarly, if $\rho_{A,B} = -1$, equation (1) implies that

$$\sigma_P^2 = (w_A\sigma_A - w_B\sigma_B)^2. \tag{3}$$

Substituting $w_A = 1 - w_B$, the previous expression can be made equal to zero if we pick

$$w_B = \frac{\sigma_A}{\sigma_A + \sigma_B}.$$

Therefore, we conclude that by picking

$$w_B = \begin{cases} \dfrac{\sigma_A}{\sigma_A - \sigma_B} & \text{if } \rho_{A,B} = 1, \\ \dfrac{\sigma_A}{\sigma_A + \sigma_B} & \text{if } \rho_{A,B} = -1, \end{cases} \tag{4}$$

and $w_A = 1 - w_B$ we can make the variance of the portfolio equal to zero.

The portfolio characterized by w_A and w_B in (4) is the global minimum variance portfolio that achieves a variance equal to zero when the two assets are perfectly correlated. In probability, a random variable with zero variance must be constant. Thus, in the absence of arbitrage opportunities, the return of this portfolio must be equal to the risk-free rate. If not, you could borrow at a cheaper rate and invest at a higher rate without risk, generating arbitrarily large profits for free. Such a riskless profit would only last briefly in competitive financial markets.

Thus, we must have that

$$r_f = \mathrm{E}(r_P) = w_A r_A + w_B r_B, \tag{5}$$

where w_B is determined by equation (4) and $w_A = 1 - w_B$. The expression in (5) says that in the absence of arbitrage opportunities, it is possible to create your own risk-free asset if you can trade two perfectly correlated assets.

These assets typically do not exist as such in financial markets, but financial institutions can create them. For example, a *call option* is a contract that gives its purchaser the right but not the obligation to *purchase* an asset at a specific date in the future for a price agreed upon today. Call options exhibit positive perfect correlation with their underlying asset over short intervals. Thus, combining the underlying asset with a call option written on it can create an overnight risk-free asset. This remarkable insight allowed Black and Scholes (1973) and Merton (1973) to derive a formula for pricing derivatives!

A *put option* gives an example of an asset exhibiting a perfect negative correlation with an asset over short periods of time. A put gives its purchaser the right but not the obligation to *sell* an asset at a specific date in the future for a price agreed upon today. Again, combining the underlying asset with a put option written on it can create a risk-free asset. The idea of synthesizing a risk-free asset out of perfectly correlated risky assets has spawned a gigantic industry of derivatives products.

Example 1 (Perfect Positive Correlation)**.** Suppose you have two risky assets A and B such that $\mu_A = 14\%$, $\mu_B = 19\%$, $\sigma_A = 20\%$, $\sigma_B = 30\%$, and $\rho_{A,B} = 1$.

We can use the expression for w_B defined in (4) to compute the implied risk-free rate. Thus, the portfolio defined by

$$w_B = \frac{0.20}{0.20 - 0.30} = -2,$$

and $w_A = 1 - (-2) = 3$ has zero variance. The implied risk-free rate is

$$r_f = 3 \times 0.14 - 2 \times 0.19 = 4\%.$$

We can verify that the variance of the portfolio is indeed zero,

$$\sigma_P = 3 \times 0.2 - 2 \times 0.3 = 0.$$

□

Example 2 (Perfect Negative Correlation)**.** Suppose you have two risky assets A and B such that $\mu_A = 13\%$, $\mu_B = -2\%$, $\sigma_A = 20\%$, $\sigma_B = 10\%$, and $\rho_{A,B} = -1$.

We can use the expression for w_B defined in (4) to compute the implied risk-free rate. The portfolio characterized by

$$w_B = \frac{0.20}{0.20 + 0.10} = 2/3,$$

and $w_A = 1/3$, has zero variance. The implied risk-free rate is

$$r_f = 1/3 \times 0.13 + 2/3 \times (-0.02) = 3\%.$$

The variance of the portfolio is indeed zero since

$$\sigma_P = 1/3 \times 0.2 - 2/3 \times 0.1 = 0.$$

□

Equation (5) can be written as $w_A R_A + w_B R_B = 0$, or

$$R_B = \pm \frac{\sigma_B}{\sigma_A} R_A, \tag{6}$$

where capital letters denote excess returns over the risk-free rate. The sign in equation (6) is the same as the correlation coefficient between the two assets. Thus, if two risky assets are perfectly correlated, their excess returns over the risk free rate must be proportional.

Creating Perfectly Correlated Assets

The converse of the previous statement is also true. Suppose you start with a risky asseet A and you create a new asset B with proportions w in A and $1 - w$ in the risk-free asset. The returns of B are described by

$$r_B = (1 - w) r_F + w r_A = r_F + w(r_A - r_F),$$

or

$$R_B = w R_A. \tag{7}$$

We show now that the excess returns of A and B are perfectly correlated with correlation 1 or -1, the sign of the correlation depending on the sign of w.[1]

To see this, we can compute

$$\begin{aligned} \text{Cov}(R_A, R_B) &= \text{Cov}(R_A, w R_A) \\ &= w \sigma_A^2. \end{aligned}$$

Thus, the correlation between A and B is

$$\rho_{A,B} = \frac{\text{Cov}(R_A, R_B)}{\sigma_A \sigma_B} = \frac{w \sigma_A^2}{\sigma_A |w| \sigma_A} = \begin{cases} 1 & \text{if } w > 0, \\ -1 & \text{if } w < 0. \end{cases}$$

[1] Remember that a negative w means that you short A to invest more in the risk-free asset.

By combining any risky asset with the risk-free rate we obtain a whole family of portfolios that are perfectly correlated, either positively or negatively, with each other.

The Investment Opportunity Set

We just saw that we can make perfectly correlated assets out of a risky asset and the risk-free rate. Take a risky asset A with expected return equal to μ_A and standard deviation equal to σ_A.

$$\mu_P = (1-w)rf + w\mu_A,$$
$$\sigma_P = |w|\sigma_A.$$

Combining the previous two expressions, we find that he investment opportunity set of combining A and the risk-free asset is given by:

$$\mu_P = \begin{cases} rf + \left(\frac{\mu_A - rf}{\sigma_A}\right)\sigma_P & \text{if } w > 0, \\ rf - \left(\frac{\mu_A - rf}{\sigma_A}\right)\sigma_P & \text{if } w < 0. \end{cases} \tag{8}$$

The previous expression describes two lines that have the same intercept equal to the risk-free rate. The slope coefficient of the lines is equal to plus/minus the Sharpe ratio of asset A, depending on whether you invest or borrow asset A. Figure 1 plots the investment opportunity set generated by asset A and the risk-free asset.

In the figure, the point B denotes a portfolio between A and the risk-free asset where $R_B = wR_A$ and $w > 1$. Thus, assets A and B are perfectly positively correlated. If there was no risk-free asset, both risky assets could be combined together to create a risk-free asset.

Example 3. Suppose you have two risky assets A and B such that $\mu_A = 15\%$, $\mu_B = 25\%$, $\sigma_A = 25\%$, $\sigma_B = 50\%$, and $\rho_{A,B} = 1$.

This would be the situation described in Figure 1. The slope coefficient of the line created by A and B is

$$SR = \frac{\mu_B - \mu_A}{\sigma_B - \sigma_A} = \frac{0.25 - 0.15}{0.50 - 0.25} = 0.40.$$

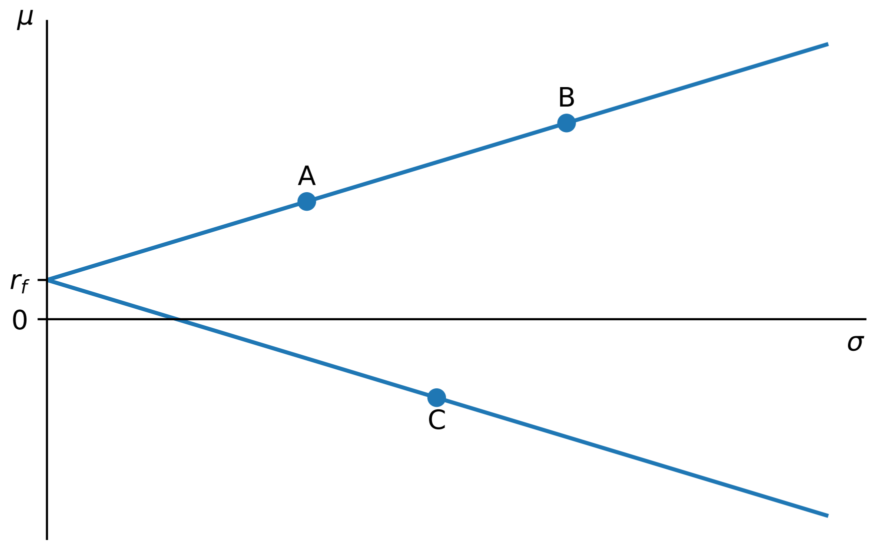

Figure 1: The figure shows the investment opportunity set of combining a risky asset A and the risk-free asset.

The line between A and B is described by

$$\mu = r_f + 0.40\sigma,$$

where r_f is the risk-free rate and represents the line intercept with the y-axis. The equation should be valid for both A and B, so we can pick either to compute the implied risk-free rate. If we use asset A we have that

$$0.15 = r_f + 0.40 \times 0.25,$$

or

$$r_f = 0.15 - 0.40 \times 0.25 = 5\%.$$

□

References

Black, Fischer, and Myron Scholes. 1973. "The Pricing of Options and Corporate Liabilities." *Journal of Political Economy* 81 (3): 637–54.

Merton, Robert C. 1973. "Theory of Rational Option Pricing." *The Bell Journal of Economics and Management Science*, 141–83.