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Forward Rates

Synthesizing Forward Rates

The term-structure of interest rates allow investors to lock future interest rates using *forward-rates* that can be synthesized from zero-coupon rates.

Denote by $Z(n)$ the price of a zero-coupon bond with face value equal to $1 and expiring at year n. Consider now two zero-coupon bonds with expirations $m < n$, that we will denote by M and N, respectively.

Let's see what happens if we sell $\frac{Z(n)}{Z(m)}$ units of bond M and invest the proceeds in bond N. The cost of buying $\frac{Z(n)}{Z(m)}$ units of M is just the units you buy times the cost per unit, i.e., $\frac{Z(n)}{Z(m)} \times Z(m) = Z(n)$. Therefore, by selling $\frac{Z(n)}{Z(m)}$ units of M you get the amount needed to buy 1 unit of bond N.

The table below describes the cash flows form this strategy, where a negative sign is used to indicate that you pay a certain amount, and a positive sign indicates that you are receiving that amount.

Year	0	m	n
Sell $\frac{Z(n)}{Z(m)}$ units of M	$Z(n)$	$-\frac{Z(n)}{Z(m)}$	0
Buy 1 unit of N	$-Z(n)$	0	1
Total	0	$-\frac{Z(n)}{Z(m)}$	1

The table shows that this strategy creates a *forward-starting* zero-coupon bond. Instead of investing a certain amount today at a known interest rate, and therefore guaranteeing a known amount at maturity, this newly created security guarantees *today* that if you invest a certain amount in year m, you will earn a known interest rate in year n.

The implicit annualized interest rate $f(m,n)$ that applies to this newly created security is called the *forward-rate* from year m to n, which must satisty

$$\frac{1}{(1+f(m,n))^{(n-m)}} = \frac{Z(n)}{Z(m)},$$

so that

$$f(m,n) = \left(\frac{Z(m)}{Z(n)}\right)^{\frac{1}{n-m}} - 1. \tag{1}$$

Example 1 (Computing Forward Rates)**.** Suppose that we have the following information on zero-coupon bonds for various maturities:

Bond	Maturity (years)	Price ($)
Z_1	1	920
Z_2	2	840
Z_3	3	760
Z_4	4	710

a. The forward rate $f(2,4)$ that applies from years 2 to 4 satisfies

$$f(2,4) = \left(\frac{840}{710}\right)^{1/2} - 1 = 8.77\%.$$

b. The forward rate $f(1,4)$ that applies from years 1 to 4 satisfies

$$f(1,4) = \left(\frac{920}{710}\right)^{1/3} - 1 = 9.02\%.$$

c. The forward rate $f(3,4)$ that applies from years 3 to 4 satisfies

$$f(3,4) = \frac{760}{710} - 1 = 7.04\%.$$

□

Sometimes, the forward rate is expressed as a function of the zero-coupon rates implicit in the zero-coupon bond prices. If we denote by r_m and r_n the zero-coupon rates corresponding to maturities m and n, respectively, Equation (1) can be re-written as

$$f(m,n) = \left(\frac{(1+r(n))^n}{(1+r(m))^m}\right)^{\frac{1}{n-m}} - 1. \tag{2}$$

Note that if we could fix the forward rate $f(m,n)$ beforehand, then it must be the case that investing from 0 to m at $r(m)$, and then rolling the deposit from m to n at $f(m,n)$, should be equivalent to investing all the way from 0 to n at $r(n)$. If not, there would be a simple arbitrage opportunity where we could invest using the strategy that delivers the most and borrow using the other alternative.

Therefore, it must be the case that:

$$(1+r(m))^m(1+f(m,n))^{n-m} = (1+r(n))^n,$$

which also delivers equation (2).

Example 2 (One-Year Forward Rates)**.** The table below shows the zero-coupon rate per year compounded annually for different maturities. The forward rate that applies from the previous to the current year is computed in the last column.

Year	Zero Rate (%)	Forward Rate (%)
1	4.0	
2	5.0	6.01
3	5.6	6.81
4	6.0	7.21
5	6.3	7.51

In the table, the forward rate that applies from year 1 to 2 is computed as

$$f(1,2) = \frac{1.05^2}{1.04} - 1 = 6.01\%,$$

whereas the forward rate that applies from year 4 to 5 is given by

$$f(1,2) = \frac{1.063^5}{1.06^4} - 1 = 7.51\%.$$

□

Theories of the Term-Structure of Interest Rates

We call the collection of YTM of zero-coupon bonds the term structure of zero-coupon bond yields or the *term structure of zero rates*. The *yield curve* usually refers to a similar concept which is the par yields of coupon bonds. It is possible to derive the term structure of zero rates from par yields. Formally, the term structure of interest rates is a function $y(n)$ that determines the zero rate for a specific maturity n. In these notes, we will express these rates per year with annual compounding.

The term structure of zero rates can take different shapes as a function of the time-to-maturity:

- upward sloping (most typical),
- downward sloping,
- flat,
- and hump shaped.

The Expectations Hypothesis

There have been many theories proposed to explain the shape of the term structure of zero rates, and among them the most popular one is the *expectations hypothesis* (EH). In a world populated by risk-neutral investors, the expected (holding period) return of investing in any asset should be the risk-free zero-rate that applies to that period.

Denote by B the price of a bond paying annual coupons C and expiring at time T. If the expectations hypothesis holds, the expected HPR of buying this bond at time 0 for B_0 and selling this bond next year for B_1 must be $r_0(1)$, the zero rate at time 0 that applies to cash flows in year 1,

$$\mathsf{E}(R_1) = \frac{\mathsf{E}(B_1) + C}{B_0} - 1 = r_0(1).$$

Note that B_1 is unkown today since we do not know how interest rates will evolve over time.

Example 3 (Computing an Expected Bond Price)**.** Suppose you have the following information on zero-coupon rates.

Maturity (years)	Rate (%)
1	5.0%
2	5.5%
3	6.0%
4	6.3%
5	6.5%

Consider a 4% annual-paying coupon bond over a notional of $1,000 expiring in 5 years. The price of this bond today can be computed by discouting its cash flows at the appropriate zero-rate.

$$B_0 = \frac{40}{1.05} + \frac{40}{1.055^2} + \frac{40}{1.06^3} + \frac{40}{1.063^4} + \frac{1040}{1.065^5} = 898.02$$

If the EH holds, the price of this bond next year, B_1, should be such that

$$\frac{B_1 + C}{B_0} = 1 + r_0(1),$$

implying $B_1 = 898.02 \times 1.05 - 40 = \$902.92$. □

Our version of the EH is consistent with assuming that expected future discount rates can be computed using forward rates, i.e.,

$$\mathsf{E}(Z_t(n)) = \mathsf{E}\left(\frac{1}{(1 + r_t(n))^n}\right) = \frac{1}{(1 + f(t, t+n))^n} \tag{3}$$

We can use equation (3) to derive an expression that relates expected future zero-rates and forward rates.

The first thing to notice is that Jensen's inequality implies that

$$\mathsf{E}\left(\frac{1}{(1+r_t(n))^n}\right) > \frac{1}{(1+\mathsf{E}(r_t(n)))^n},$$

which combined with (3) delivers

$$\mathsf{E}(r_t(n)) > f(t, t+n).$$

Therefore, our version of the EH implies that forward rates are a downward biased estimate of future zero-rates. The difference between the two, however, is in general very small. To see why, consider the function

$$h(x) = \frac{1}{(1+x)^n}.$$

The second order Taylor expansion of $h(x)$ around $x = \mu$ is:

$$\frac{1}{(1+x)^n} \approx \frac{1}{(1+\mu)^n} - n\frac{(x-\mu)}{(1+\mu)^{n+1}} + \frac{1}{2}n(n+1)\frac{(x-\mu)^2}{(1+\mu)^{n+2}}. \tag{4}$$

Consider now a random variable X such that $\mathsf{E}(X) = \mu$ and $\mathsf{V}(X) = \sigma^2$. Taking expectations on both sides of (4) shows that

$$\mathsf{E}\left(\frac{1}{(1+X)^n}\right) \approx \frac{1}{(1+\mu)^n} + \frac{1}{2}n(n+1)\frac{\sigma^2}{(1+\mu)^{n+2}}. \tag{5}$$

Denoting $f = f(t, t+n)$, $\mu = \mathsf{E}(r_t(n))$ and $\sigma^2 = \mathsf{V}(r_t(n))$, we can combine (3) and (5) to obtain

$$\frac{1+\mu}{1+f} \approx \left(1 + \frac{1}{2}n(n+1)\left(\frac{\sigma}{1+\mu}\right)^2\right)^{1/n}.$$

The previous expression shows that the difference between expected future rates and forward rates depends on a *convexity adjustment* term on the right of the above expression. For typical values of μ and σ, the convexity adjustment term is very small, which allows us to conclude that if the EH holds, forward rates are good forecasts of expected future zero-rates, i.e.,

$$\mathrm{E}(r_t(n)) \approx f(t, t+n).$$

In order to understand the implications of the EH for the shape of the term structure of zero rates, consider the evolution of the one-year rate over time. Remember that $r_n(1)$ denotes the one-year zero-rate in n years from now.

If the EH holds, equation (2) implies that

$$1 + \mathrm{E}(r_n(1)) \approx \frac{(1 + r_0(n+1))^{n+1}}{(1 + r_0(n))^n} = (1 + r_0(1)) \left(\frac{1 + r_0(n+1)}{1 + r_0(1)} \right) \left(\frac{1 + r_0(n+1)}{1 + r_0(n)} \right)^n$$

If the term structure is upward sloping, i.e., $r_0(1) < r_0(n) < r_0(n+1)$, one-year zero-rates are expected to increase until the term structure is flat. The opposite should occur if the term structure is downward sloping. Therefore, the EH implies that the term structure of zero rates should be flat on average.

The Liquidity Preference Theory

Empirically, we observe that the term-structure of interest rates is most of the time upward sloping. Thus, the EH does not seem to explain the term-structure of interest rates well in practice. One potential pitfall of the EH is that it basically assumes that all zero coupon bonds are perfect substitutes, independent of their maturity.

When investors are *risk averse*, they care not only about the expected short rate but also about its volatility. Investors in long-term bonds want to be compensated for committing their funds for a long time since they face price risk uncertainty if they need to sell before maturity. Conversely, issuers of bonds are willing to pay a higher interest rate on long-term bonds because they can lock in an interest rate for many years. Thus, the liquidity preference theory implies that:

$$r_0(2) > \frac{r_0(1) + \mathrm{E}(r_1(2))}{2}.$$

Segmented Markets Theory

This theory is also known as the *preferred habitat theory*. According to this theory, some investors only trade short-term bonds implying that short-term interest rates are solely determined by supply and demand among these investors. Other investors, like insurance companies, only trade long-term bonds, implying that long-term interest rates are determined by supply and demand among these investors. This view may explain why 30 year rates are typically lower than 20 year rates.