Bond Pricing

Pricing a Bond Paying Annual Coupons

An annual-paying coupon-bond pays a periodic amount C every year, and its principal or face-value F at maturity, as shown in the figure below.

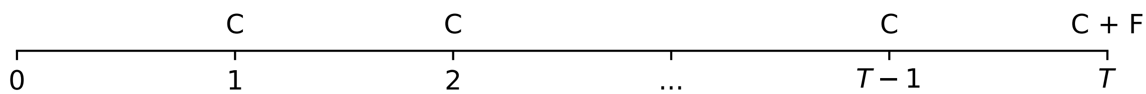

Figure 1: The figure shows the cash flows of an annual-paying coupon bond.

Usually, the coupon is expressed as a percentage of the face value of the bond. The ratio C/F is called the coupon rate and is expressed as a percent. Also, most bonds are issued in \$1,000 denominations in which case we have $F = \$1{,}000$.

The constant discount rate that prices the bond correctly is called the yield-to-maturity (YTM). For valuation purposes, in the case of annual paying coupon bonds it is customary to express the YTM per year compounded annually. If we denote by y the YTM per year compounded annually, we have that:

$$
\begin{aligned}
B &= \frac{C}{(1+y)} + \frac{C}{(1+y)^2} + \cdots + \frac{C}{(1+y)^{T-1}} + \frac{C+F}{(1+y)^T} \\
&= \frac{C}{y}\left(1 - \frac{1}{(1+y)^T}\right) + \frac{F}{(1+y)^T},
\end{aligned}
$$

where in the second line I have used the formula for the present value of an annuity.[1] Alternatively, financial calculators can perform these computations easily.

Example 1 (Computing a Bond Price)**.** Consider a bond paying annual coupons with a coupon rate of 5% per year over a principal of $1,000 and maturity 30 years. The YTM is 6% per year with annual compounding. To compute the bond's price, we could use a financial calculator:

	N	I/Y	PV	PMT	FV
Given:	30	6		50	1000
Solve for:			-862.35		

The price is then $862.35. The negative sign provided by the financial calculator represents the fact that if we pay $862.35 today, we are entitled to be paid every year $50 for 30 years, and $1,000 at the end of 30 years. □

In order to compute the YTM, we need to find the discount rate that gives the quoted price of the bond.

Example 2 (Computing a YTM)**.** Consider a bond paying annual coupons with a coupon rate of 6.5% per year over a principal of $1,000 and maturity 25 years. The bond trades for $1,020. To compute the YTM, we could use a financial calculator:

	N	I/Y	PV	PMT	FV
Given:	25		-1020	65	1000
Solve for:		6.34			

The YTM is then 6.34% per year with annual compounding. □

[1] If the interest rate is r per period, the present value of an annuity paying C every period for N periods is:

$$PV = \frac{1}{r}\left(1 - \frac{1}{(1+r)^N}\right).$$

Most bonds are issued at or close to par value. That is, the coupon rate is chosen to be the par yield of the bond. As time passes, though, the price of the bond might increase if yields decrease or decrease if yields increase. Since the YTM is hard to compute, practitioners like to compute the *current yield* which is defined as the coupon rate divided by the quoted price.

A bond is said to trade at a *premium* if its flat price is greater than its face value. Conversely, a bond is said to trade at a *discount* if the quoted price is less than par value.

Example 3 (Current Yield vs YTM)**.** Consider a 3-year, annual paying coupon bond, with coupon rate of 8% and face value of $1,000. The table below displays the current yield and YTM for different values of the bond price.

	Price	Coupon rate	Current Yield	YTM
Premium Bond	1,100	8%	7.27%	4.37%
Par bond	1,000	8%	8.00%	8.00%
Discount Bond	900	8%	8.89%	12.18%

We can see that current yield is quite different from the YTM unless the bond trades at par. □

In general, the current yield is not equal to the YTM, as the previous example shows. The YTM equals the current yield only when the bond is a perpetuity. Therefore, for long maturity bonds the current yield is a good approximation of the YTM.

Example 4 (Current Yield of a Long Term Bond)**.** Consider a 30-year, annual paying coupon bond, with coupon rate of 8% and face value of $1,000. The table below displays the current yield and YTM for different values of the bond price.

	Price	Coupon rate	Current Yield	YTM
Premium Bond	1,100	8%	7.27%	7.18%
Par bond	1,000	8%	8.00%	8.00%

	Price	Coupon rate	Current Yield	YTM
Discount Bond	900	8%	8.89%	8.97%

We can see that for a long term bond the current yield approximates the YTM well. ☐

Using Zero Rates to Price a Bond

In general, bonds with different maturities and coupon rates will have different YTM. The standard way to harmonize this problem is to use zero rates to price each cash flow at the right maturity and use the resulting price to compute the YTM.

In these notes, we will denote by $Z_t(n)$ the price of a zero coupon bond at time t expiring at time $t+n$.

Example 5. The table below presents zero rates for different maturities expressed per year with annual compounding.

Maturity (years)	1	2	3	4	5	6
Zero Rate (%)	2.0	3.0	3.5	4.0	4.3	4.5

a. The price of a zero-coupon bond that pays $1,000 in 3 years is given by

$$Z_0(3) = \frac{1000}{1.035^3} = \$901.94.$$

b. The price of a zero-coupon bond that pays $1,000 in 5 years is given by

$$Z_0(5) = \frac{1000}{1.043^5} = \$810.17.$$

c. The price of a zero-coupon bond that pays $1,000 in 6 years is given by

$$Z_0(6) = \frac{1000}{1.045^6} = \$767.90.$$

□

We can use the zero rates of Example 5 to compute the price and YTM of a coupon bond.

Example 6. Consider a bond paying annual coupons with a coupon rate of 4% per year over a principal of $1,000 and maturity 6 years. The table below presents zero rates for different maturities expressed per year with annual compounding.

Maturity (years)	1	2	3	4	5	6
Zero Rate (%)	2.0	3.0	3.5	4.0	4.3	4.5

The bond pays a coupon of $40 every year for six years, and the principal at the end. To compute the price of the bond, we need to add the discounted cash flows at the relevant interest rate.

Maturity (years)	1	2	3	4	5	6
Cash flow	40	40	40	40	40	1040
Discounted Cash Flow	39.22	37.70	36.08	34.19	32.41	798.61

The bond price is then the sum of the discounted cash flows and equal to $978.21. We can now use a financial calculator to compute the bond's YTM.

	N	I/Y	PV	PMT	FV
Given:	6		-978.21	40	1000
Solve for:		4.42			

The bond's YTM is 4.42% per year. □

Holding Period Return

The *holding period return* (HPR) is the net return of purchasing a bond over a period of time. It is usually expressed in annualized form. For a bond that pays coupons annually, the HPR at time t is defined as:

$$R_{t+1} = \frac{B_{t+1} + C}{B_t} - 1,$$

where B_t denotes the price of the bond at time t, B_{t+1} denotes the price of the bond a period later just after paying its coupon C. The HPR is not known at time t since we do not know the price of the bond a period later.

When you buy a bond, the return on your investment or realized return is equal to the YTM if and only if you can re-invest the coupons at the same rate and you hold the bond until maturity. In most cases, the HPR will be different from the YTM because you must re-invest the coupons at a different rate, or you sell the bond before maturity at a price that corresponds to a different yield-to-maturity.

Example 7. Suppose that a 3-year zero-coupon bond with face value $1,000 has a YTM of 5% per year compounded annually. The bond's current price is:

$$Z_0(3) = \frac{1000}{(1.05)^3} = \$863.84.$$

a. If next year the YTM changes to 7%, the new bond price will be:

$$Z_1(2) = \frac{1000}{(1.07)^2} = \$873.44.$$

Notice that the zero-coupon bond next year has two years until maturity. The realized (holding period) return over the one year period is then:

$$R_1 = \frac{873.44}{863.84} - 1 = 1.11\%.$$

b. If the YTM in year 1 remains at 5%, the new price is

$$Z_1(2) = \frac{1000}{(1.05)^2} = \$907.03,$$

and the realized return over the one year period becomes

$$R_1 = \frac{907.03}{863.84} - 1 = 5.00\%.$$

□

Example 8. A 4-year coupon bond has face value \$1,000, coupons paid every year of \$80, and a YTM of 8% per year with annual compounding. Since the YTM equals the coupon rate, the bond trades at par, that is $B_0 = \$1,000$.

a. If the coupons can be reinvested at 8% per year, the bond price next year will be also par value, providing an annual return of 8% per year.

b. If next year the interest rate falls to 4%, the new bond price will be:

$$B_1 = \frac{80}{0.04}\left(1 - \frac{1}{1.04^3}\right) + \frac{1000}{1.04^3} = \$1{,}111.00.$$

The realized (holding period) return over the one year period is then:

$$R_1 = \frac{1111 + 80}{1000} - 1 = 19.10\%.$$

□

Pricing a Bond Paying Semi-Annual Coupons

Most bonds in the U.S. such as Treasuries pay coupons twice per year. If the coupon rate of the bond is expressed per year, then the semi-annual coupon payment is equal to half the annual coupon rate times the par-value of the bond. The YTM of semi-annual paying coupon bonds is usually quoted as per year with semi-annual compounding.

Example 9. Consider a bond paying semi-annual coupons with a coupon rate of 5% per year over a principal of $1,000 and maturity 30 years. The YTM is 6% per year with semi-annual compounding. To compute the bond's price, we could use a financial calculator as follows:

	N	I/Y	PV	PMT	FV
Given:	60	3		25	1000
Solve for:			-861.62		

Note that to solve for the bond price we use 60 semi-annual periods, the annual YTM is divided by 2 since it is compounded semi-annually and the semi-annual payment is 2.5% of the face value. The price is then $861.62. □

The *par yield* of a bond is the coupon rate that makes the bond trade at par.

Example 10. Consider a bond paying semi-annual coupons with a coupon rate of C% per year over a principal of $1,000 and maturity 3 years. The table below presents zero rates for different maturities expressed per year with annual compounding.

Maturity (years)	0.5	1	1.5	2	2.5	3
Zero Rate (%)	2.0	3.0	3.5	4.0	4.3	4.5

The par yield C is such:

$$\frac{C/2}{1.02^{0.5}} + \frac{C/2}{1.03^{1}} + \frac{C/2}{1.035^{1.5}} + \frac{C/2}{1.04^{2}} + \frac{C/2}{1.043^{2.5}} + \frac{100 + C/2}{1.045^{3}} = 100.$$

If we denote

$$A = \frac{1}{1.02^{0.5}} + \frac{1}{1.03^{1}} + \frac{1}{1.035^{1.5}} + \frac{1}{1.04^{2}} + \frac{1}{1.043^{2.5}} + \frac{1}{1.045^{3}}$$

and $D = \frac{1}{1.045^3}$, the par yield solves

$$100 = \frac{C}{2}A + 100D.$$

Thus,

$$C = \frac{2(100 - 100D)}{A} = 4.41.$$

The 3-year par yield for a semi-annual paying coupon bond is therefore 4.41% per year. □

The figure below shows in the inverse relationship between bond prices and yields for a 30-year bond with a 5% coupon rate making semi-annual coupon payments. We can see that if the YTM is less than 6% the bond trades at a premium whereas if the YTM is greater than 6% the bond trades at a discount. If the YTM is equal to 6% then the bond trades at par.

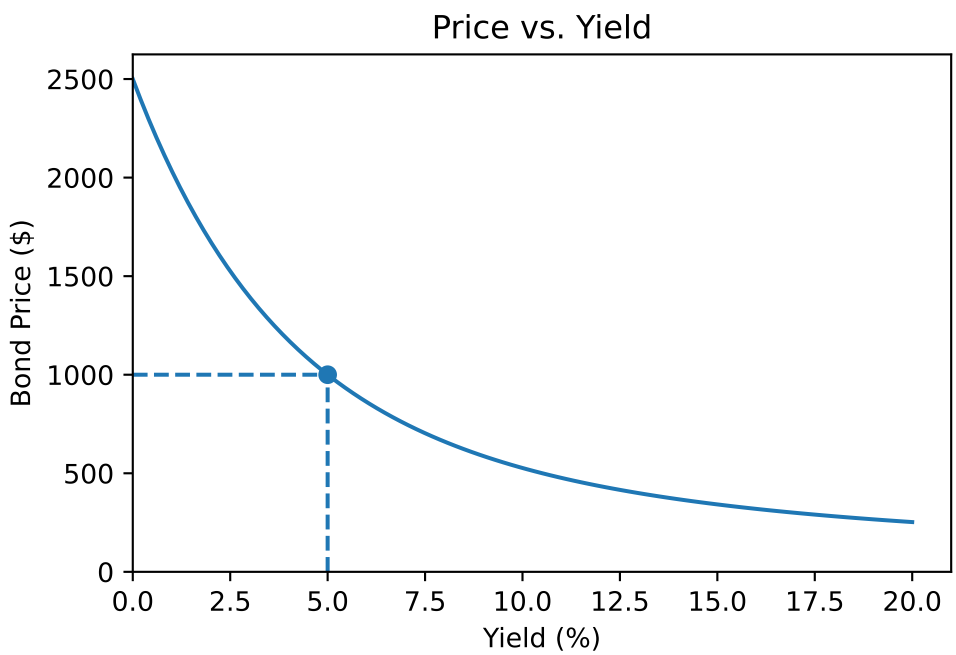

Figure 2: The figure shows the price of a 30-year bond paying semi-annual coupons of 5% per year over a notional of $1,000, as a function of the YTM.

Figure 2 also shows that the inverse relationship between bond price and yields is not linear but convex. The convexity of the bond price implies that interest rates volatility has a positive effect in expected bond returns.

Flat vs. Invoice Prices

So far we have assumed that the next coupon comes exactly in six or twelve months, just after a coupon payment has been made. In general, if a bond is purchased between coupon payments, we need to take into account of the exact timing of the cash flows in order to compute the *invoice price*.

Let D be the number of days between coupon payments, and denote by d the number of days since the last coupon payment. Then, for a bond paying k coupons per year and having N coupons left[2], paying an annual coupon of C and face value F we must have that:

$$\begin{aligned}\text{Invoice Price} &= \frac{C/k}{(1+y/k)^{1-d/D}} + \frac{C/k}{(1+y/k)^{2-d/D}} + \cdots + \frac{C/k+F}{(1+y/k)^{N-d/D}} \\ &= \left[\frac{C/k}{y/k}\left(1 - \frac{1}{(1+y/k)^N}\right) + \frac{F}{(1+y/k)^N}\right] \times (1+y/k)^{d/D},\end{aligned}$$

where y is the YTM of the bond expressed per year and compounded k times per year.

As the bond approaches maturity, however, the invoice price approaches par value. This is knows as the *pull-to-par* effect.

The *accrued interest* of the bond is defined as:

$$\text{Accrued Interest} = \frac{C}{k} \times \frac{d}{D}.$$

The *quoted* or *flat price* of the bond is then defined as the difference between the invoice price and the accrued interest:

$$\text{Flat Price} = \text{Invoice Price} - \text{Accrued Interest}.$$

Example 11. Consider a bond with a coupon rate of 8% per year, paying semi-annual coupons over a notional of $1,000. If 30 days have elapsed since the last coupon payment,

[2] For example, if a bond pays semi-annual coupons and the bond expires in 9.8 years, then there are 20 coupons left.

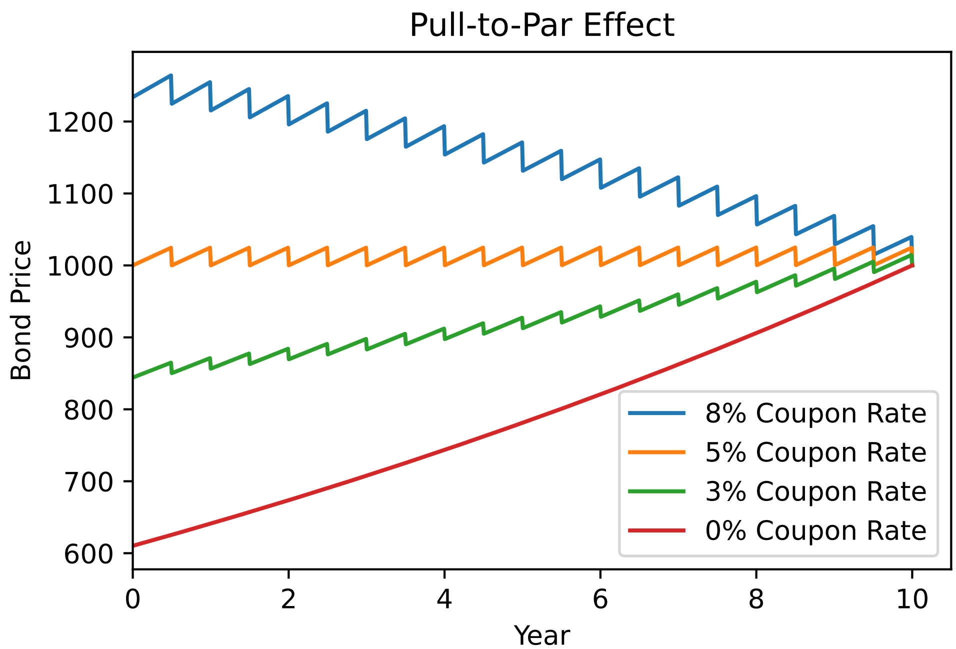

Figure 3: The figure shows the evolution of the invoice price of a 10-year semi-annual paying coupon bond for different values of the coupon rate assuming that the YTM stays at 5% per year with semi-annual compounding.

and there are 182 days in the semi-annual coupon period, the accrued interest on the bond is $40 \times (30/182) = \$6.59$. If the quoted price on the bond is \$995, then the invoice price will be $995 + 6.59 = \$1{,}001.59$. □

The Excel formula `PRICE` computes the flat price using the above expressions. The `PRICE` function syntax has the following arguments:

- *Settlement*: The security's settlement date. The security settlement date is the date after the issue date when the security is traded to the buyer.
- *Maturity*: The security's maturity date. The maturity date is the date when the security expires.
- *Rate*: The security's annual coupon rate. It has to be entered as a percent or decimal.
- *Yld*: The security's annual yield. It has to be entered as a percent or decimal.
- *Redemption*: The security's redemption value per \$100 face value. The Excel formula assumes that the coupon rate is paid over a notional of \$100. Therefore, to price a bond that pays the face value in full we set the redemption equal to 100.
- *Frequency*: The number of coupon payments per year. For annual payments, frequency = 1; for semiannual, frequency = 2; for quarterly, frequency = 4.
- *Basis*: The type of day count basis to use.

Basis	Day count basis
0 or omitted	US (NASD) 30/360
1	Actual/actual
2	Actual/360
3	Actual/365
4	European 30/360

This function provides the flat price according to the street convention, which simplifies the timing of the cash flows regardless of whether they fall on a workday or a holiday. The typical day count for Treasury securities is Actual/Actual whereas for corporate bonds is 30/360.

Example 12. Consider a Treasury Note for which we have the following information:

- Settlement date: 7/21/2017
- Maturity date: 5/15/2027
- Coupon rate: 2.375%
- Yield-to-maturity: 2.400%

Using the above parameters, and a redemption of 100, frequency of 2 and basis equal to 1, the Excel function `PRICE` gives a flat price of $99.78084174. □