Topics in Quantitative Finance | Prof. Lorenzo Naranjo
FIN 5018 | Fall 2025

Problem Set 4

Instructions: This problem set is due on 9/26 at 11:59 pm CST and is an individual assignment. All problems must be handwritten. Scan your work and submit a PDF file.

Problem 1. Consider a non-dividend paying stock whose returns follow a diffusion such that

$$\frac{dS}{S} = \mu dt + \sigma dB,$$

where B is a one-dimensional Browninan motion. The risk-free rate is denoted by r. If Λ is a stochastic discount factor such that

$$\frac{d\Lambda}{\Lambda} = -rdt - \frac{\lambda}{\sigma} dB,$$

what should be the sign of $\left(\frac{d\Lambda}{\Lambda}\right)\left(\frac{dS}{S}\right)$ if $\mu < r$?

Problem 2. Suppose that you have two independent Brownian motions B_1 and B_2. How can you build a new Brownian motion B_3 using B_1 and B_2 so that $(dB_2)(dB_3) = \rho dt$.

Problem 3. Suppose the SDF is given by

$$\frac{d\Lambda}{\Lambda} = -rdt - \lambda_1 dB_1 - \lambda_2 dB_2 - \dots - \lambda_n dB_n,$$

where $B_1, B_2, \dots, B_n$ are independent Brownian motions. What does it mean if one of the λ's is equal to zero?

Problem 4. Consider two assets whose price processes are given by

$$\frac{d\mathbf{S}}{\mathbf{S}} = \boldsymbol{\mu} dt + \boldsymbol{\sigma} d\mathbf{z},$$

where $d\mathbf{z}$ is a vector of three independent Brownian motions B_1, B_2, and B_3. You know that

$$\boldsymbol{\sigma} = \begin{pmatrix} 0.3 & -0.1 & 0.2 \\ 0.15 & 0.2 & -0.05 \end{pmatrix}.$$

a. Compute the instantaneous correlation between the returns of each asset.
b. Find a Brownian motion $B_4 = a_1 B_1 + a_2 B_2 + a_3 B_3$ whose increments are independent from the instantaneous returns of the two assets.

Problem 5. Consider a stochastic discount factor in continuous time given by

$$\frac{d\Lambda}{\Lambda} = -r_f dt - \lambda_1 dB_1 - \lambda_2 dB_2,$$

where B_1 and B_2 are independent Brownian motions. Suppose that you have two non-dividend paying assets with the following dynamics:

$$\frac{dS_1}{S_1} = \mu_1 dt + \sigma_1 dB_1,$$

and

$$\frac{dS_2}{S_2} = \mu_2 dt + \sigma_{21} dB_1 + \sigma_{22} dB_2.$$

Suppose that $r_f = 0.05$, $\mu_1 = 0.15$, $\mu_2 = 0.20$, $\sigma_1 = 0.3$, $\sigma_{21} = 0.5$, and $\sigma_{22} = -0.1$.

a. Compute the instantaneous correlation between the returns of each asset.
b. Determine λ_1 and λ_2.

Problem 6. In this problem we consider a probability space $(\Omega, \mathcal{F}, \mathrm{P})$ in which time goes from 0 to T. Consider two non-dividend paying assets S_1 and S_2 that follow geometric Brownian motions

$$\frac{dS_1}{S_1} = \mu_1 dt + \sigma_1 dB_1,$$
$$\frac{dS_2}{S_2} = \mu_2 dt + \sigma_2 dB_2,$$

where B_1 and B_2 are two potentially correlated Brownian motions such that $dB_1 dB_2 = \rho_{1,2} dt$. There is a stochastic discount factor Λ so that each process ΛS_i is a local martingale for $i = 1, 2$. The SDF is characterized by

$$\frac{d\Lambda}{\Lambda} = -rdt - \lambda dB,$$

where B is a Brownian motion correlated with B_1 and B_2 so that $dBdB_i = \rho_i$ for $i = 1, 2$, and λ is the market price of risk of B. The money-market account is denoted by β and grows geometrically at the risk-free rate r so that

$$\frac{d\beta}{\beta} = rdt.$$

a. Show that
$$\lambda = \frac{\mu_i - r}{\rho_i \sigma_i} \text{ for } i = 1, 2.$$
Explain why $\rho_i = 1$ for efficient assets.

b. Let P^* denote the risk-neutral measure defined by
$$\frac{d\,\mathrm{P}^*}{d\,\mathrm{P}} = \mathcal{E}_T,$$
where $\mathcal{E} = \Lambda\beta$. Compute B_1^* and B_2^* so that both terms are Brownian motions under P^*.

c. Compute the instantaneous correlation between B_1^* and B_2^*.

d. If you were to form a zero-cost portfolio in which you go long S_1 and you go short S_2, what should be the risk-neutral drift of such portfolio?