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The Stochastic Discount Factor

In this note I generalize the notion of stochastic discount factor by showing that it emerges naturally as a consequence of the law of one price. The absence of arbitrage opportunities implies the existence of a strictly positive discount factor. Thus, in complete markets there is a unique strictly positive stochastic discount factor. Original sources of the analysis can be found in Hansen and Richard (1987), Hansen and Jagannathan (1991) and Hansen and Jagannathan (1997).

The Set of Traded Payoffs

In this economy, not all payoffs are necessarily traded unless the market is *complete*. We denote by X the linear subspace of traded payoffs spanned by $\{x_1, x_2, \dots, x_N\}$, where $N \leq S$ and all payoffs are assumed to be linearly independent. Denote by

$$\mathbf{x}' = (x_1, x_2, \dots, x_N) \tag{1}$$

a vector containing all the basis payoffs. In the following, we assume that the Gram matrix

$$\mathrm{E}(\mathbf{x}\mathbf{x}') = \sum_{s=1}^{S} q(s)\mathbf{x}(s)\mathbf{x}'(s)$$

is invertible.

We can create other payoffs by buying or selling our N original assets. Any $x \in X$ can be expressed as:

$$x = \sum_{i=1}^{N} a_i x_i, \tag{2}$$

for $a_i \in \mathbb{R}$, $1 \leq i \leq N$. Finally, we denote by π_i the price of asset i for $1 \leq i \leq N$, and by

$$\pi' = (\pi_1, \pi_2, \dots, \pi_n)$$

a vector containing the prices. of the N basis payoffs.

The Law of One Price

Now, we would like to know if there is a way to create a pricing functional $p : X \to \mathbb{R}$ that gives the price of any traded payoff. Clearly, we have that $p(x_i) = \pi_i$ for $1 \leq i \leq N$, and our intuition tells us that the price of any other asset should be given in terms of the other asset prices. If this was not the case, this would be an arbitrage.

Example 1. Suppose that $p(x) = 1$ and $p(y) = 2$. There is also an asset $z = 3x + 4y$ such that $p(z) = 12$. Is there an arbitrage opportunity?

Of course! We could buy 3 units of x and 4 units of y and bundle them as z. The cost of the bundle is \$11, but we can sell it for \$12, generating a riskless profit of \$1 per trade. Since there is no shortage of these securities, we could continue doing it until the prices of x and y go up and/or the price of z goes down. □

In order to avoid these type of situations, we will assume the following.

Assumption 1 (The Law of One Price). *Suppose that $x_i \in X$ and $a_i \in \mathbb{R}$ for $i \in 1, 2, \dots, N \leq S$. If $x = \sum_{i=1}^{N} a_i x_i \in X$, then*

$$p(x) = \sum_{i=1}^{N} a_i p(x_i). \tag{3}$$

In competitive markets, the **law of one price** (LOOP) guarantees that the price of a basket of stocks is equal to the sum of the prices of its constituents. This logic is at the heart of how **Exchange-Traded Funds** (ETF) operate, as the next example shows.

Example 2. An ETF is a type of investment fund that is traded on stock exchanges, similar to individual stocks. ETFs hold a diversified portfolio of assets, such as stocks, bonds, or commodities, which provides investors with broad exposure to specific markets or investment strategies.

ETF arbitrage is the mechanism that helps keep the market price of an ETF in line with its Net Asset Value (NAV). Authorized Participants (APs), typically large financial institutions, have the ability to create or redeem ETF shares in large blocks called creation units.

When the ETF market price is higher than the NAV, APs can buy the underlying securities of the ETF in the open market and then deliver them to the ETF issuer in exchange for new ETF shares. The AP can then sell these ETF shares at the higher market price, making a profit. This buying of underlying securities pushes their prices up, while the selling of new ETF shares pushes the ETF price down, bringing the two prices closer together.

When the ETF market price is lower than the NAV, APs can buy ETF shares in the open market and deliver them to the ETF issuer in exchange for the underlying securities. The AP can then sell these underlying securities at the higher NAV price, making a profit. This buying of ETF shares pushes their price up, while the selling of the underlying securities pushes their prices down, again bringing the two prices closer together.

This creation and redemption process happens continuously and helps to keep the ETF price in line with the NAV. The arbitrage opportunities are typically small but sufficient for APs to engage in the process for profit, ensuring that the ETF price does not deviate significantly from its NAV.

You can find more information here. □

The fact that the market for ETFs is so liquid and works flawlessly reassures us that LOOP is a reasonable axiom to start working from. The law of one price implies the price functional defined in (3) is a continuous linear functional, and the Riesz representation theorem implies the existence of a stochastic discount factor x^* that is also a payoff in X.

Therefore, it must be the case that

$$x^* = \sum_{i=1}^{N} c_i x_i = \mathbf{c}'\mathbf{x},$$

such that $p(x_i) = \mathrm{E}(x^* x_i)$ for $1 \leq i \leq N$. Thus,

$$\pi' = \mathrm{E}(x^* \mathbf{x}) = \mathrm{E}(\mathbf{c}'\mathbf{x}\mathbf{x}') = \mathbf{c}'\,\mathrm{E}(\mathbf{x}\mathbf{x}'),$$

or

$$\mathbf{c}' = \pi' \, \mathrm{E}(\mathbf{x}\mathbf{x}')^{-1}.$$

We can now verify that the payoff

$$x^* = \pi' \, \mathrm{E}(\mathbf{x}\mathbf{x}')^{-1}\mathbf{x}$$

is a valid discount factor. Take an arbitrary payoff $x = \mathbf{x}'\mathbf{a} \in X$, and compute

$$\begin{aligned} \mathrm{E}(x^* x) &= \mathrm{E}(\pi' \, \mathrm{E}(\mathbf{x}\mathbf{x}')^{-1}\mathbf{x}\mathbf{x}'\mathbf{a}) \\ &= \pi' \, \mathrm{E}(\mathbf{x}\mathbf{x}')^{-1} \, \mathrm{E}(\mathbf{x}\mathbf{x}')\mathbf{a} \\ &= \pi'\mathbf{a} \\ &= p(x), \end{aligned}$$

which shows that x^* is a valid stochastic discount factor.

Property 1. *Given a linear subspace $X \in L$ of traded payoffs spanned by $\{x_1, x_2, \dots, x_N\}$, where $N \leq S$, the vector*

$$x^* = \pi' \, \mathrm{E}(\mathbf{x}\mathbf{x}')^{-1}\mathbf{x} \in X \tag{4}$$

is a valid stochastic discount factor.

Any other SDF m will price the assets correctly, so that

$$\mathrm{E}((m - x^*)x) = \mathrm{E}(mx) - \mathrm{E}(x^* x) = p(x) - p(x) = 0.$$

This shows that we can create new SDFs by combining x^* with any vector e orthogonal to $\mathcal{X}$. In other words, all the SDFs that price assets correctly in X can be written as:

$$m = x^* + e,$$

where e x for all $x \in X$.

The Principle of No-Arbitrage

A violation to the law of one price is an arbitrage opportunity, but not all arbitrage opportunities are violations of the law of one price. There might be situations in which some investors manage to build a payoff that is positive in some states and zero in others. In competitive financial markets, the price of that payoff must be positive, otherwise the demand for that asset would be infinite.

Assumption 2 (Principle of No-Arbitrage)**.** *The price of a payoff that is positive in all states and strictly positive in at least one state of the world must be positive.*

The principle of no-arbitrage (PNA) is a stronger assumption than the LOOP, as the next property shows.

Property 2.

$$\mathit{PNA} \Rightarrow \mathit{LOOP}.$$

Proof

We will prove this claim by contradiction. Assume that NA holds but not LOOP. Then LOOP implies that the price of a zero payoff must be zero. Without loss of generality, a violation of LOOP implies that $p_0 = p(0) > 0$.

Say that we have a payoff x^+ that is positive in some states of the world and zero otherwise, and whose price is $p > 0$. Form a portfolio that buys one unit of x^+ and sells $n > \frac{p}{p_0}$ units of the zero payoff. The cost of that portfolio is

$$\pi = p - np_0 < p - \frac{p}{p_0}p_0 = 0,$$

but its payoff is positive in some states and zero otherwise, a contradiction. □

A more important consequence of PNA is that it implies the existence of a strictly positive SDF, i.e., a SDF that is greater than zero in all states. The reverse is also true. There are no arbitrage opportunities if there is one SDF that is strictly positive.

Property 3.

$$\mathit{PNA} \Leftrightarrow \exists m > 0.$$

See page 4 in Duffie (2010) for a proof of this result. The proof uses the separating hyperplane theorem to separate the cone of strictly positive payoffs from the subspace of traded assets.

Implementing the Model with Data

We can associate for any random variable $y \in L$ the vector of payoffs

$$\mathbf{y} = \begin{pmatrix} y(1) \\ y(2) \\ \vdots \\ y(S) \end{pmatrix}$$

characterizing the value of y in each state $s = 1, \dots, S$. Therefore, we have a one-to-one correspondence between the random variable $y : \mathcal{S} \to \mathbb{R}$ and the vector $\mathbf{y} \in \mathbb{R}^S$ since the s-th coordinate of $\mathbf{y}$ corresponds to the value of y in state s, i.e.,

$$\mathbf{y}_s = y(s),$$

for all $s = 1, \dots, S$.

Denote by

$$\mathbf{X} = \begin{pmatrix} \mathbf{x}(1) \\ \mathbf{x}(2) \\ \vdots \\ \mathbf{x}(N) \end{pmatrix}$$

the $S \times N$ matrix whose rows are the values of the random vector $\mathbf{x}$ defined in (1) in each state of the world. We can also interpret $\mathbf{X}$ as the matrix whose columns are the vector of payoffs of each basis asset.

Arrow-Debreu Securities

The payoff space L is complete and can be spanned by a basis of S linearly independent vectors. Define the random variable e_s as

$$e_s(i) = \begin{cases} 1 & \text{if } i = s, \\ 0 & \text{otherwise.} \end{cases}$$

Thus, any payoff $x \in L$ can be written as

$$x = x(1)e_1 + x(2)e_2 + \cdots + x(N)e_N, \tag{5}$$

since in state s the random variable x is equal to $x(s)$ and the random variable e_s pays 1.

Maximum Sharpe Ratio

Since $\mathrm{E}(mR) = 1$, we have that

$$1 = \mathrm{E}(mR) = \mathrm{E}(m)\,\mathrm{E}(R) + \mathrm{Cov}(m, R).$$

Therefore,

$$\begin{aligned} \mathrm{E}(R^i) - R^f &= -\frac{\mathrm{Cov}(m, R^i)}{\mathrm{E}(m)} \\ &= -\frac{\rho(m, R^i)\sigma(m)\sigma(R^i)}{\mathrm{E}(m)} \end{aligned}$$

Since $|\rho| \leq 1$ we have that:

$$\left| \frac{\mathrm{E}(R^i) - R^f}{\sigma(R^i)} \right| \leq \frac{\sigma(m)}{\mathrm{E}(m)} \tag{6}$$

References

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