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The Geometry of the Payoff Space

In this note I describe the mathematical structure of the payoff space that we will use to characterize the space of traded payoffs and stochastic discount factors. Even though the results described in this note apply to infinite dimensional Hilbert spaces, we will restrict our attention to the study of finite dimensional Euclidean spaces.

Probability Structure

Uncertainty is represented by a finite set $\mathcal{S} = \{1, \dots, S\}$ of states, defining a finite probability space $(\mathcal{S}, \pi)$. The set of all random variables defined in $\mathcal{S}$ is denoted by L and is called the **payoff space**. Thus, for any $x \in L$ we have that the vector $(x(1), x(2), \dots, x(S)) \in \mathbb{R}^S$ defines all the possible payoffs in each state, and the probability of getting a payoff in a particular state is given by $\Pr(x = x(s)) = \pi(s)$ for all $s \in S$. We assume throughout that $\pi(s) > 0$ for all $s \in \mathcal{S}$, that is, we will not consider possible outcomes that happen with probability zero.

The payoff space is clearly a linear vector space since for any $x, y \in L$ and $\alpha, \beta \in \mathbb{R}$ we have that $\alpha x + \beta y \in L$. We endow the payoff space with an inner product $\langle \cdot, \cdot \rangle : L \times L \to \mathbb{R}$ defined such that for any $x, y \in L$, we have that

$$\langle x, y \rangle = \mathrm{E}(xy) = \sum_{s=1}^{S} \pi(s) x(s) y(s).$$

In finite-dimensional spaces, we can use the inner product to define the Euclidean norm $\|\cdot\| : L \to \mathbb{R}^+$ for all $x \in L$ as

$$\|x\| = \sqrt{\langle x, x \rangle}.$$

Clearly, $\|x\| = 0 \Leftrightarrow x = 0$. Therefore, the second moment of x defined as $\|x\|^2 = \mathrm{E}(x^2)$ plays an important role since it allows us to asses the *convergence* of a series of payoffs towards a certain payoff.

Projections

Given $x, y \in L$, consider the vectors $y_x = \alpha x$ and $z = y - y_x$. We say that y_x is the projection of y on the subspace generated by $\{x\}$ if the norm of z is minimal. To obtain the projection, we need to compute the α that minimizes $\|z\| = \|y - \alpha x\| = \mathrm{E}(y - \alpha x)^2$. The first-order condition of this problem is:

$$0 = \mathrm{E}((y - \alpha x)x) = \langle y - \alpha x, x\rangle = \langle z, x\rangle,$$

which implies that $\alpha = \dfrac{\langle x, y\rangle}{\langle x, x\rangle}$ and $\langle z, y_x\rangle = 0$.

We say that two vectors $x, y \in L$ are orthogonal if their inner product is equal to zero. Thus, we have that $y_x \perp z$, implying that the vector y can be decomposed into two orthogonal components. Indeed, we have that

$$\|y\|^2 = \|z + y_x\|^2 = \|z\|^2 + 2\langle z, y_x\rangle + \|y_x\|^2 = \|z\|^2 + \|y_x\|^2,$$

which is a generalization of the classical Pythagorean theorem.

Property 1 (Orthogonal Decomposition)**.** *Given $x, y \in L$, the projection of y on the subspace generated by $\{x\}$ is given by $y_x = \dfrac{\langle x, y\rangle}{\langle x, x\rangle}x$. The vector $z = y - y_x$ is orthogonal to y_x, implying that*

$$\|y\|^2 = \|z\|^2 + \|y_x\|^2. \tag{1}$$

Equation (1) implies that $\|y\|^2 \geq \|y_x\|^2$, with equality occurring whenever y is proportional to x. Therefore, we have that

$$\|y\|^2 \geq \|y_x\|^2 = \left\|\frac{\langle x, y\rangle}{\langle x, x\rangle}x\right\|^2 = \frac{\langle x, y\rangle^2}{\|x\|^2}.$$

The previous expression is known as the Cauchy-Schwartz inequality and is fundamental in the study of Euclidean vector spaces.

Property 2 (Cauchy-Schwartz Inequality). *Given $x, y \in L$ we have that*

$$|\langle x, y\rangle| \leq \|x\|\|y\|. \tag{2}$$

Linear Functionals

Given $x, y \in L$ and $\alpha, \beta \in \mathbb{R}$, a linear functional $f : L \to \mathbb{R}$ satisfies

$$f(\alpha x + \beta y) = \alpha f(x) + \beta f(y).$$

We say that the linear functional $f : L \to \mathbb{R}$ is bounded if

$$|f(x)| \leq M\|x\|$$

for all $x \in L$. In other words, the absolute value of the functional cannot grow infinitely for a finite x. A bounded linear functional is also called a continuous linear functional. The smallest M for which this inequality remains true is called the norm of f, i.e.,

$$\|f\| = \inf\{M : |f(x)| \leq M\|x\|, \text{ for all } x \in L\}.$$

For a given $m \in L$ and any $x \in L$, the functional

$$f(x) = \langle m, x\rangle = \mathrm{E}(mx) = \sum_{s=1}^{S} \pi(s)m(s)x(s)$$

is linear since

$$f(\alpha x + \beta y) = \mathrm{E}(m(\alpha x + \beta y)) = \alpha\,\mathrm{E}(mx) + \beta\,\mathrm{E}(my) = \alpha f(x) + \beta f(y).$$

Furthermore, the Cauchy-Schwartz inequality implies that

$$|f(x)| = |\langle m, x\rangle| \leq \|m\|\|x\|,$$

showing that the linear functional f is bounded and hence continuous. Since the previous inequality is an equality whenever x is proportional to m, we have that $\|m\|$ is the smallest bound of f, showing that $\|f\| = \|m\|$.

Conversely, consider a linear functional $f : L \to \mathbb{R}$. The set $K = \{x \in L : f(x) = 0\}$ describes a hyperplane that can be described by a normal vector z. Thus, $\langle x, z \rangle = 0$ for all $x \in K$. Wihtout loss of generality, assume that z has been appropriately scaled so that $f(z) = 1$.

Given any $x \in L$, we have that $x - f(x)z \in K$ since $f(x - f(x)z) = f(x) - f(x)f(z) = 0$. Moreover, $z \perp K$, implying that

$$0 = \langle x - f(x)z, z \rangle = \langle x, z \rangle - f(x)\langle z, z \rangle.$$

The previous expression implies that

$$f(x) = \frac{\langle x, z \rangle}{\langle z, z \rangle} = \langle x, m \rangle,$$

where $m = \dfrac{z}{\|z\|^2}$. The previous analysis is an important result known as the *Riesz representation theorem*.

Property 3 (Riesz Representation Theorem). *If $f : L \to \mathbb{R}$ is a bounded linear functional, there exists a unique vector $m \in L$ such that for all $x \in L$, $f(x) = \langle m, x \rangle$. Furthermore, we have $\|f\| = \|y\|$ and every m determines a unique bounded linear functional.*