

Consumption Based Asset Pricing

Introduction

Most of modern asset pricing can be cast in terms of the **stochastic discount factor**. The idea is to find the price p_t at time t of a payoff x_{t+1} paid at time $t + 1$. The payoff will in general be random, and therefore unknown at time t .

For example, if you purchase a stock at time t your payoff at time $t + 1$ will be the price p_{t+1} at which you can sell the stock plus possibly a dividend d_{t+1} , or $x_{t+1} = p_{t+1} + d_{t+1}$.

Example 1. Consider a stock that currently trades for 7. The table below shows the dividend and price expected next period for different scenarios.

	Probability	Dividend	Price	Payoff
Boom	0.3	1.0	10	11.0
Normal	0.5	0.8	7	7.8
Recession	0.2	0.7	4	4.7

The payoff of purchasing the stock today is therefore a random variable defined in some finite probability space (Ω, P) . Note that the expected payoff is 8.14, which implies an expected return of 16.29%.

The big question in finance is then how to find the price p_t of the asset at time t . In Example 1, the payoffs of the stock are generic and could have been the payoffs of any type of asset. Thus, the asset pricing theory we will develop in these notes applies not only to stocks, but also to bonds and derivatives.

The notes below follow closely Chapter 1 in Cochrane (2009).

Pricing Assets

Property 1. *The price of an asset that pays x_{t+1} at time $t + 1$ is given by*

$$p_t = E_t(m_{t+1}x_{t+1}), \quad (1)$$

where

$$m_{t+1} = \delta \frac{u'(c_{t+1})}{u'(c_t)}, \quad (2)$$

is called the stochastic discount factor or pricing kernel.

A simple way to model the tradeoff between consuming today versus tomorrow is to write

$$U(c_t, c_{t+1}) = u(c_t) + \delta E_t[u(c_{t+1})], \quad (3)$$

where $E_t[\cdot]$ denotes the expectation conditional on the information available at time t , $u(\cdot)$ is an increasing and concave function of consumption, and $\delta < 1$ is a discounting factor.

In the expression above, $U(c_t, c_{t+1})$ is the value function of consuming c_t today and c_{t+1} tomorrow. The investor must decide then how much to consume today, how much to save, and how to invest the savings among different assets. We assume that the investor has already arranged her investments so that she can consume optimally at time t and $t + 1$. Thus, the investor has already decided how to invest in all assets available to her.

Consider now the case in which the investor deviates from her optimal investment rule and decides to purchase ξ additional shares of an asset that currently trades for p_t . Her consumption at time t drops to $c_t - \xi p_t$ since she needs the money to purchase the additional shares, whereas her consumption at time $t + 1$ changes to $c_{t+1} + \xi x_{t+1}$ since she will receive an additional payoff from the asset she just purchased.

Given the original levels of consumption c_t and c_{t+1} , the new utility can be seen as a function of ξ ,

$$U(\xi) = u(c_t - \xi p_t) + \delta E_t[u(c_{t+1} + \xi x_{t+1})].$$

We can now try to maximize the previous function and find the optimal number of shares to purchase of the asset. The first-order condition (FOC) for the optimal ξ must satisfy

$$U'(\xi) = -p_t u'(c_t - \xi p_t) + \delta E_t[x_{t+1} u'(c_{t+1} + \xi x_{t+1})] = 0.$$

However, if the original levels of consumption c_t and c_{t+1} are already optimal, then we know that $\xi = 0$ is indeed the optimum, implying that

$$-p_t u'(c_t) + \delta E_t[x_{t+1} u'(c_{t+1})] = 0,$$

or

$$p_t = E_t \left[\delta \frac{u'(c_{t+1})}{u'(c_t)} x_{t+1} \right]. \quad (4)$$

Equation (4) is the fundamental asset pricing formula. Intuitively, the price of an asset is high if the asset pays well when marginal utility is high, that is, when consumption is low. An asset that pays well when consumption is high and not much when consumption is low is not attractive for a risk-averse investor, carrying a low price.

The term

$$m_{t+1} = \delta \frac{u'(c_{t+1})}{u'(c_t)}$$

is called the **stochastic discount factor** (SDF) or **pricing kernel**. The stochastic discount factor captures all the information needed to price assets given their payoffs. We will see later that we can extend the formula to price multi-period cash flows both in discrete and continuous time.

To simplify notation, we will typically write the pricing equation as

$$p = E(mx), \quad (5)$$

where is understood that p denotes the price today of a payoff x paid next period.

If x is a random variable defined on a finite probability space (Ω, P) , the pricing equation can be written as

$$p = \sum_{\omega \in \Omega} P(\omega) m(\omega) x(\omega), \quad (6)$$

where ω denotes a state of the world, $P(\omega)$ is the probability of the outcome ω occurring, and $x(\omega)$ is the payoff if ω happens.

Example 2. Consider an investor with a power utility function

$$u(c) = \frac{c^{1-\gamma}}{1-\gamma}.$$

The stochastic discount factor is then given by

$$m_{t+1} = \delta \left(\frac{c_{t+1}}{c_t} \right)^{-\gamma}.$$

The investor's current consumption is $c_t = 6.5$, and is considering investing in two assets X and Y . The table below presents the probabilities of different scenarios, along with the future consumption and payoffs of the assets.

	Probability	Consumption	Payoff X	Payoff Y
Boom	0.3	9.0	9.8	6.0
Normal	0.5	6.7	8.3	5.0
Recession	0.2	5.4	6.5	7.1

If $\gamma = 4$ and $\delta = 0.95$, we can compute the stochastic discount factor (SDF) for each scenario as

	Probability	SDF
Boom	0.3	0.258
Normal	0.5	0.842
Recession	0.2	1.994

Therefore, we can use equation (6) to compute the prices of X and Y . We find that $p(x) = 6.85$ and $p(y) = 5.4$, respectively. The expected return of each asset is equal to the expected payoff divided by its price minus one. Thus, $E(r^x) = 22.57\%$ and $E(r^y) = 5.91\%$.

So far we have been silent about the currency used to price assets and quantify payoffs. It is clear, however, that our derivation of the fundamental pricing equation (4) used units of real consumption to quantify prices and payoffs. The stochastic discount factor defined in (2) is therefore a real discount factor.

If the Euler equation holds in real terms (consumption), does it also hold in nominal terms (dollars)? The answer is yes, and we can define a nominal discount factor that works with different currencies.

Let $p_t = p_t^*/\Pi_t$ and $x_{t+1} = x_{t+1}^*/\Pi_{t+1}$ where Π_t denotes the price level and we use asterisks to denote nominal quantities. Then,

$$\frac{p_t^*}{\Pi_t} = E_t \left(m_{t+1} \frac{x_{t+1}^*}{\Pi_{t+1}} \right).$$

The previous expression implies that

$$p_t^* = E_t (m_{t+1}^* x_{t+1}^*),$$

where $m_{t+1}^* = m_{t+1} \frac{\Pi_t}{\Pi_{t+1}}$. We obtain the same equation as before but now expressed in nominal terms. In the analysis, m_{t+1}^* plays the role of a nominal discount factor.

Prices and Returns

We will use

$$R_{t+1} = \frac{x_{t+1}}{p_t}$$

to denote the gross rate of return of investing in the stock. For example, if an investment of \$100 generates \$105, the gross return is $R = 1.05$, whereas the net return is $r = 5\%$.

We can divide (1) by p on both sides to get

$$1 = E(mR). \tag{7}$$

Therefore, for any security i we always have

$$E(mR^i) = 1. \quad (8)$$

A risk-free asset pays next period $x = 1$ no matter what, so that its return R^f is constant. This implies that $E(mR^f) = 1$, or

$$R^f = \frac{1}{E(m)}.$$

The stochastic discount factor has all the information to recover the behavior of interest rates in the economy.

Example 3. Using the data of Example 2, we find that $E(m) = 0.8972$. Therefore,

$$R^f = \frac{1}{0.8972} = 1.1146,$$

or a net return of 11.46% per period.

A zero-cost portfolio involves buying asset i and shorting asset j , generating a zero-cost return $R^e = R^i - R^j$. Equation (8) implies that

$$E(mR^e) = 0$$

for any zero-cost portfolio. The price of a zero-cost portfolio is of course zero, since it involves no cash outflow to create it. The payoffs, however, need not be equal to zero as they are determined by $R^i - R^j$. Since there is no price to pay initially, at least in theory, this zero-cost return can be amplified arbitrarily.

Compensation for Time and Risk

The pricing of risky cash flows should incorporate two dimensions. On the one hand, cash flows paid in the future should be discounted to account for the time value of money. On the other hand, riskier payoffs should be generate lower prices.

To analyze these issues, we can again start from (1)

$$\begin{aligned} p &= E(mx) = E(m) E(x) + \text{Cov}(m, x) \\ &= \frac{E(x)}{R^f} + \text{Cov}(m, x). \end{aligned} \quad (9)$$

We can see in the previous expression that the first term represents the time value of money. Expected cash flows should be discounted to account for the time-value of money. The second term in the expression is a risk adjustment. Thus, the price of the asset is high when R^f is low and/or the covariance with the stochastic discount factor is high.

We can use returns instead of prices to write (1) for asset i as

$$1 = E(mR^i) = E(m) E(R^i) + \text{Cov}(m, R^i).$$

Dividing the previous expression by $E(m)$, we find that

$$E(R^i) - R^f = -\frac{\text{Cov}(R^i, m)}{E(m)} = \beta_{i,m} \lambda_m, \quad (10)$$

where $\beta_{i,m} = \frac{\text{Cov}(R^i, m)}{V(m)}$ and $\lambda_m = -\frac{V(m)}{E(m)} < 0$.

Equation (10) defines a typical beta pricing model. Here, the source of systematic risk is exposure to the discount factor. The beta in equation (10) captures how sensitive the security is to the stochastic discount factor, and is specific to the security. That's why its subscript depends on both i and m .

The lambda in equation (10) is the price of risk of the discount factor. Note that in this case we have that $\lambda < 0$. This is because an asset with a high beta pays well in bad times, that is, when marginal utility is high. Therefore, agents like that asset pushing its price up, generating a lower expected return. To achieve lower expected returns when beta is high and higher expected returns when beta is low, the price of risk has to be negative.

Equation (2) implies that $\beta_{i,m} = \frac{\text{Cov}(R^i, u'(c_{t+1}))}{V(u'(c_{t+1}))}$ and $\lambda_m = -\frac{V(u'(c_{t+1}))}{E(u'(c_{t+1}))}$. Thus, in the beta pricing model defined by (10) only the marginal utility of future consumption matters

to price assets. This is a simplified version of the consumption CAPM of Breeden (1979) and Lucas (1978).

Multi-Period Asset Pricing

Consider now a stream of consumption $\{c_{t+j}\}_{j=0}^{\infty}$. We can extend the utility function defined earlier the following way:

$$V_t = E_t \sum_{j=0}^{\infty} \beta^j u(c_{t+j}). \quad (11)$$

Note that we can also write (11) as

$$\begin{aligned} V_t &= u(c_t) + \beta E_t \sum_{j=0}^{\infty} \beta^j u(c_{t+1+j}) \\ &= u(c_t) + \beta E_t E_{t+1} \sum_{j=0}^{\infty} \beta^j u(c_{t+1+j}) \\ &= u(c_t) + \beta E_t V_{t+1}. \end{aligned} \quad (12)$$

Equation (12) is the corresponding Bellman equation for equation (11).

Now consider now a stream of dividends $\{D_{t+j}\}_{j=1}^{\infty}$. The same perturbation analysis can be used to show that

$$p_t = E_t \sum_{j=1}^{\infty} \beta^j \frac{u'(c_{t+j})}{u'(c_t)} D_{t+j}. \quad (13)$$

Therefore, the corresponding stochastic discount factor to price a dividend paid at time $t + j$ is:

$$m_{t+j} = \beta^j \frac{u'(c_{t+j})}{u'(c_t)}.$$

We can use (13) to price any asset with cash flows paid at time $T > 1$. For example, consider a real discount bond expiring with time-to-maturity time T and face value 1 unit

of consumption. The price of the bond is given by

$$B_t = E_t(m_{t+T}).$$

Similarly, the price of a nominal discount bond expiring with time-to-maturity time T and face value \$1 is

$$B_t^* = E_t\left(m_{t+T} \frac{\Pi_t}{\Pi_{t+T}}\right) = E_t(m_{t+T}^*).$$

Finally, the price of a call option with strike K and maturity T is given by

$$c_t = E_t(m_{t+T}^*(S_{t+T} - K)^+).$$

In conclusion, the stochastic discount factor framework is universal and allows for the pricing of all securities. We will see later that the only requirement for the existence of a stochastic discount factor is the absence of arbitrage opportunities.

References

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