The Greeks

Delta Hedging

In many cases (but not always), the seller of an option might want to hedge a position dynamically, i.e. by re-balancing a portfolio consisting in the stock and a risk-free bond. Even though static hedging might be desirable (like buying another option), it might not be feasible or economically viable. In such cases, it is useful to think about how to replicate the option dynamically. We call this process delta hedging, and the resulting portfolio is said to be *delta neutral*.

Delta of a Portfolio

The delta of an option (or portfolio) measures its sensitivity to the underlying asset price. Formally, if we denote by V the value of a portfolio (possibly containing the stock, risk-free bonds and/or options), the delta of the portfolio is defined as:

$$\Delta = \frac{\partial V}{\partial S}$$

The delta of a European call or put option can be computed from the Black-Scholes formula as shown before. The process of *delta hedging* involves buying or selling stocks as determined by the delta.

Example 1. Consider a non-dividend paying stock that currently trades at \$50. A trader just sold 50 call option contracts (5,000 options) written on the stock. The current option price is \$4.13 and the option's delta is 0.591. The money-market risk-free rate is 5% per year with simple compounding.

1. *First hedge*: The delta of the position is $-5{,}000 \times 0.591 = -2{,}955$. To delta hedge the position, the trader buys 2,955 shares for a cost of \$147,750. By writing the calls, the trader receives $5{,}000 \times 4.13 = \$20{,}650$, which amounts to a net expense of \$127,100 that the trader borrows. Since the trader is delta neutral, the interest rate on the loan is the risk-free rate.
2. *Price Change*: During the next week, the stock price increases to \$50.53, the option price increases to \$4.35, and the delta changes to 0.612. The delta of the option position changes to $-5{,}000 \times 0.612 = -3{,}060$.

3. *Profit & Loss*: The long stock position is now worth $2{,}955 \times 50.53 = \$149{,}316.15$, whereas the short call position is worth $-5{,}000 \times 4.35 = -\$21{,}750$. The loan now accrues to $-127{,}100\left(1 + \frac{0.05}{52}\right) = -\$127{,}222.21$. The weekly P&L is then $149,316.15 - 21,750 - 127,222,21 = \$343.94$, which is a small percentage of the total exposure. To keep things simple we will keep the (small) hedging errors in a *separate account*.
4. *Hedge re-balancing*: The trader buys an additional $3{,}060 - 2{,}955 = 105$ shares to maintain delta neutrality. The total cost of the new long position in shares is $3{,}060 \times 50.53 = \$154{,}621.80$. The trader needs to borrow in total $154{,}621.80 - 21{,}750 = \$132{,}871.80$, or an additional $132{,}871.80 - 127{,}222,21 = \$5{,}649.59$.

To maintain delta neutrality, we must continue the same process week after week until the expiration of the options. □

Example 2. Consider a non-dividend paying stock that currently trades at $50. The money-market risk-free rate is 5% per year with continuous compounding, and the volatility of log-returns is 25% per year. Compute the delta of a European call option with maturity 6 months and strike of $50.

$$d_1 = \frac{\ln(50/50) + (0.05 + \frac{1}{2}(0.25)^2)(0.5)}{0.25\sqrt{0.5}} = 0.2298$$

The delta is then equal to $\Phi(d_1) = 0.591$. □

Example 3. Consider:

1. A long position in 100,000 call options (1,000 contracts) with strike price $100 and expiration in 9 months. The delta of each option is 0.597.
2. A short position in 200,000 call options (2,000 contracts) with strike $110 and expiration in 6 months. The delta of each option is 0.368.
3. A short position in 50,000 put options (500 contracts) with strike $90 and expiration in 3 months. The delta of each option is –0.162.

The delta of the portfolio is:

$$100{,}000 \times 0.597 - 200{,}000 \times 0.368 - 50{,}000 \times (-0.162) = -5{,}800$$

The portfolio can then be made delta neutral by *buying* 5,800 shares. □

Delta, Theta and Gamma

Theta of a Portfolio

The theta (Θ) of a portfolio (V) captures the rate of change in value of that portfolio with respect to the passage of time, i.e.

$$\Theta = \frac{\partial V}{\partial t}$$

The theta is also referred as the *time-decay* of the portfolio and is usually monitored by traders since it is a good proxy for gamma in a delta neutral portfolio. For European call and put options we have that:

$$\Theta_C = -e^{-\delta T}\frac{S\,\Phi^{'}(d_1)\sigma}{2\sqrt{T}} - rKe^{-rT}\,\Phi(d_2) + \delta Se^{-\delta T}\,\Phi(d_1)$$
$$\Theta_P = -e^{-\delta T}\frac{S\,\Phi^{'}(d_1)\sigma}{2\sqrt{T}} + rKe^{-rT}\,\Phi(-d_2) - \delta Se^{-\delta T}\,\Phi(-d_1)$$

Gamma of a Portfolio

The gamma (Γ) of a portfolio measures the rate of change of the portfolio's delta with respect to the price of the underlying asset, i.e.

$$\Gamma = \frac{\partial \Delta}{\partial S} = \frac{\partial^2 V}{\partial S^2}$$

When gamma is small, delta changes slowly and the portfolio is kept delta neutral without many changes. On the other hand, when gamma is high, it is important to monitor the portfolio frequently and adjust delta neutrality as needed.

The gamma for European call and put options is:

$$\Gamma_C = \Gamma_P = \frac{e^{-\delta T}\,\Phi^{'}(d_1)}{S\sigma\sqrt{T}} = \frac{Ke^{-rT}\,\Phi^{'}(d_2)}{S^2\sigma\sqrt{T}}$$

Gamma and Delta Neutrality

Remember that according to Ito's Lemma:

$$dV = \frac{\partial V}{\partial S}dS + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}(dS)^2 + \frac{\partial V}{\partial t}dt$$

or

$$dV = \Delta dS + \frac{1}{2}\Gamma(dS)^2 + \Theta dt$$

Therefore, in a delta-neutral portfolio we have that:

$$dV = \frac{1}{2}\Gamma(dS)^2 + \Theta dt$$

which implies that:

$$\Delta V \approx \frac{1}{2}\Gamma(\Delta S)^2 + \Theta\Delta t$$

Example 4. Suppose that the gamma of a delta-neutral portfolio is 10,000. A jump of +\$2 or −\$2 in the underlying asset will approximately increase the value of the portfolio by $\frac{1}{2}10{,}000 \times 2^2 = \$20{,}000$. □

Example 5. A trader's portfolio is delta-neutral and has a gamma of 5,000. The delta and gamma of a traded call option is 0.52 and 1.60, respectively. The trader wants to make the portfolio both delta and gamma-neutral.

The portfolio can be made gamma-neutral by selling $\frac{5{,}000}{1.60} = 3{,}125$ call options. The portfolio can now be made delta-neutral by buying $0.52 \times 3{,}125 = 1{,}625$ shares of the underlying asset. Note that the shares have zero gamma, so they do not change the gamma of the portfolio but only affect its delta. □

Example 6. Let's compute the gamma of the option in Example 2. We compute first

$$d_1 = \frac{\ln(50/50) + (0.05 + \frac{1}{2}(0.25)^2)(0.5)}{0.25\sqrt{0.5}} = 0.2298.$$

We can now compute

$$\Phi'(d_1) = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}(0.2298)^2} = 0.3885.$$

Finally, the gamma of the option is given by:

$$\Gamma = \frac{\Phi'(d_1)}{S\sigma\sqrt{T}} = \frac{0.1550}{50(0.25)\sqrt{0.5}} = 0.0440$$

□

Gamma and Theta

Remember the fundamental Black-Scholes differential equation:

$$(r-\delta)S\frac{\partial V}{\partial S}+\frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2}+\frac{\partial V}{\partial t}=rV$$

which can be re-written using the Greeks as:

$$(r-\delta)S\Delta+\frac{1}{2}\sigma^2 S^2\Gamma+\Theta=rV$$

Hence, for a delta-neutral portfolio we have that:

$$\frac{1}{2}\sigma^2 S^2\Gamma+\Theta=rV$$

Therefore, when theta is large and positive, the gamma of a portfolio tends to be large and negative, and vice-versa.

Vega and Rho

Vega

The volatility that gives the right price of the option under the Black-Scholes is called the implied volatility. The sensitivity of the option to its implied volatility is called vega:

$$\mathcal{V}=\frac{\partial V}{\partial \sigma}$$

Usually vega risk is more relevant for longer maturity options, whereas gamma risk is more prominent for shorter maturity options.

$$\mathcal{V}_C=\mathcal{V}_P=Se^{-\delta T}\,\Phi'(d_1)\sqrt{T}=Ke^{-rT}\,\Phi'(d_2)\sqrt{T}$$

Example 7. Consider a portfolio that is delta neutral, with a gamma of -5,000 and a vega of -8,000. The options shown in the table below can be traded.

Table 1: The Greeks of the Portfolio

	Delta	Gamma	Vega
Original Portfolio	0	-5,000	-8,000
Option 1	0.6	0.5	2.0
Option 2	0.5	0.8	1.2

The portfolio can be made vega neutral by including a long position in 4,000 of Option 1. This would increase delta to 2,400 and require that 2,400 units of the asset be sold to maintain delta neutrality. The gamma of the portfolio would change from -5,000 to -3,000.

To make the portfolio both gamma and vega neutral, both Option 1 and Option 2 can be used. We must then solve:

$$-5{,}000 + 0.5N_1 + 0.8N_2 = 0$$
$$-8{,}000 + 2.0N_1 + 1.2N_2 = 0$$

These yields $N_1 = 400$ and $N_2 = 6{,}000$. The new delta is $400 \times 0.6 + 6{,}000 \times 0.5 = 3{,}240$. Hence, we sell 3,240 units of the underlying asset to maintain delta-neutrality. □

Rho

The rho (ρ) of a portfolio measures the rate of change of the portfolio's value with respect to the risk-free rate, i.e.

$$\rho = \frac{\partial V}{\partial r}$$

The rho for a European call and put options is:

$$\rho_C = KTe^{-rT}\,\Phi(d_2) > 0$$
$$\rho_P = -KTe^{-rT}\,\Phi(-d_2) < 0$$

The rho of a call is positive since it is a levered position in the risk-free asset whereas the rho of the put is negative since it borrows the stock to invest in the risk-free asset.

Summary

To summarize, in the Black-Scholes model where

$$dS = (r - \delta)Sdt + \sigma S dW$$

we have the following results for European call and put options.

Table 2: The Greeks on a Dividend Paying Asset

Variable	Call	Put
V	$Se^{-\delta T}\,\Phi(d_1) - Ke^{-rT}\,\Phi(d_2)$	$Ke^{-rT}\,\Phi(-d_2) - Se^{-\delta T}\,\Phi(-d_1)$
Δ	$e^{-\delta T}\,\Phi(d_1)$	$-e^{-\delta T}\,\Phi(-d_1)$
Γ	$\frac{e^{-\delta T}\,\Phi'(d_1)}{S\sigma\sqrt{T}} = \frac{Ke^{-rT}\,\Phi'(d_2)}{S^2\sigma\sqrt{T}}$	
Θ	$rV - (r-\delta)S\Delta - \frac{1}{2}\sigma^2 S^2 \Gamma$	
$\mathcal{V}$	$Se^{-\delta T}\,\Phi'(d_1)\sqrt{T} = Ke^{-rT}\,\Phi'(d_2)\sqrt{T}$	
ρ	$KTe^{-rT}\,\Phi(d_2)$	$-KTe^{-rT}\,\Phi(-d_2)$

Practice Problems

Solutions to all problems can be found at lorenzonaranjo.com/fin451.

Problem 1. What does it mean to assert that the delta of a call option is 0.7? How can a short position in 1,000 options be made delta neutral when the delta of each option is 0.7?

Problem 2. Calculate the delta of an at-the-money six-month European call option on a non-dividend paying stock when the risk-free interest rate is 10% per annum and the stock price volatility is 25% per annum.

Problem 3. What does it mean to assert that the theta of an option position is -0.1 when time is measured in years? If a trader feels that neither a stock price nor its implied volatility will change, what type of option position is appropriate?

Problem 4. What is meant by the gamma of an option position? What are the risks in the situation where the gamma of a position is large and negative and the delta is zero?

Problem 5. Suppose that a stock price is currently \$20 and that a call option with an exercise price of \$25 is created synthetically using a continually changing position in the stock. Consider the following two scenarios:

a. Stock price increases steadily from \$20 to \$35 during the life of the option
b. Stock price oscillates wildly, ending up at \$35

Which scenario would make the synthetically created option more expensive? Explain your answer.

Problem 6. A financial institution has the following portfolio of over-the-counter options on the euro:

Type	Position	Delta of option	Gamma of option	Vega of option
Call	-1,000	0.5	2.2	1.8
Call	-500	0.8	0.6	0.2
Put	-2,000	-0.4	1.3	0.7
Call	-500	0.7	1.8	1.4

A traded option is available with a delta of 0.6, a gamma of 1.5, and a vega of 0.8. Please answer the following:

a. What position in the traded option and the euro, would make the portfolio both gamma neutral and delta neutral?
b. What position in the traded option and the euro, would make the portfolio both vega neutral and delta neutral?
c. Suppose that a second traded option with a delta of 0.1, a gamma of 0.5, and a vega of 0.6 is available. How could the portfolio be made delta, gamma, and vega neutral?

Problem 7. Which of the following options strategies have **negative theta**? Assume that all options are European and written on an asset that does not pay dividends and that the risk-free rate is positive. Select all alternatives that are correct.

a. A long straddle with a strike price equal to the current stock price.
b. A long strangle with strike prices $K_1 < K_2$ in which the current stock price is between K_1 and K_2.
c. A bull spread with strike prices $K_1 < K_2$ in which the current stock price is less than K_1.
d. A bear spread with strike prices $K_1 < K_2$ in which the current stock price is less than K_1.

Problem 8. Consider a butterfly strategy with strike prices $K_1 < K_2 < K_3$ that you made using European call options written on a non-dividend-paying asset. The risk-free rate is 5% per year. In which scenarios will the gamma of the butterfly be negative? Select all that apply.

a. The current stock price is equal to K_1.
b. The current stock price is equal to K_2.
c. The current stock price is equal to K_3.
d. The gamma of a butterfly is never negative.

Problem 9. Consider a bull spread that you made by buying a European call with a strike K_1 and selling a European call with a strike K_2, where $K_1 < K_2$. Both options are written on a non-dividend-paying asset. The risk-free rate is 5% per year. Which alternative is correct?

a. The delta of the bull spread is always positive.
b. The delta of the bull spread is always negative.
c. The delta of the bull spread is negative if the stock price is less than K_1 and negative otherwise.
d. The delta of the bull spread is negative if the stock price is higher than K_2 and negative otherwise.

Problem 10. Which of the following instruments and/or strategies have a gamma equal to zero? Select all that apply.

a. A risk-free zero-coupon bond.
b. A long call and a long put with the same strike.
c. A stock.
d. A long call and a short put with the same strike.