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Properties of European Options

Introduction

In this chapter we analyze properties that must hold for European options written on a non-dividend paying asset in the absence of arbitrage opportunities. A key observation to derive many of these properties is that an option strategy with a positive payoff for all values of the underlying asset must have a positive price. Indeed, if this was not the case we would have a simple arbitrage opportunity: buy as many of these strategies as possible. You would get money today and potentially you would get also money in the future. A strategy like this could not survive in a competitive market.

The payoffs of a covered call, a protective put, or a butterfly are all positive and hence implies that the price of each of these strategies must be positive. On the other hand, a bull spread built with puts has a payoff that is negative. In such a case, the absence of arbitrage opportunities implies that its price must be negative, i.e. that you would get paid if you took that position.

In this chapter we also analyze the effect of negative interest rates on option prices. For many years, academics and practitioners thought that it was reasonable to assume that interest rates should stay positive at all times. A violation of this non-negative bound was associated with an arbitrage opportunity. The fact that interest rates were negative for almost a decade in many developed economies requires us to rethink this assumption and analyze what happens when interest rates are negative. We will see that negative interest rates have some unexpected consequences for some standard results in option pricing.

We finalize this chapter by discussing how different variables such as the current stock price, the strike price, time-to-maturity, volatility and the risk-free rate should affect the value of European call and put options written on an asset does not pay dividends.

Put-Call Parity

Building a Covered Call

Consider European call and put options with strike K and maturity T written on a non-dividend paying stock. There is also a zero-coupon risk-free bond with face value K and same maturity as the options.

Strategy A: Long stock and short call

$$\text{Cost} = S - C$$
$$\text{Payoff} = \begin{cases} S_T & \text{if } S_T \leq K \\ K & \text{if } S_T > K \end{cases}$$

Strategy B: Long bond and short put

$$\text{Cost} = Ke^{-rT} - P$$
$$\text{Payoff} = \begin{cases} S_T & \text{if } S_T \leq K \\ K & \text{if } S_T > K \end{cases}$$

The figure below shows the payoffs of both strategies.

Since both strategies have the same payoff, they should have the same price. Otherwise, there is an arbitrage opportunity that could be exploited.

> Put-Call Parity
>
> For European options written on non-dividend paying stocks, following relationship known as put-call parity must hold:
>
> $$S - C = Ke^{-rT} - P$$

Example 1. Consider a non-dividend paying stock trading at $110 and assume that the continuously-compounded risk-free rate is 5% per year. A European call option with strike price $110 and maturity 9 months trades for $13.30. Then, according to put-call

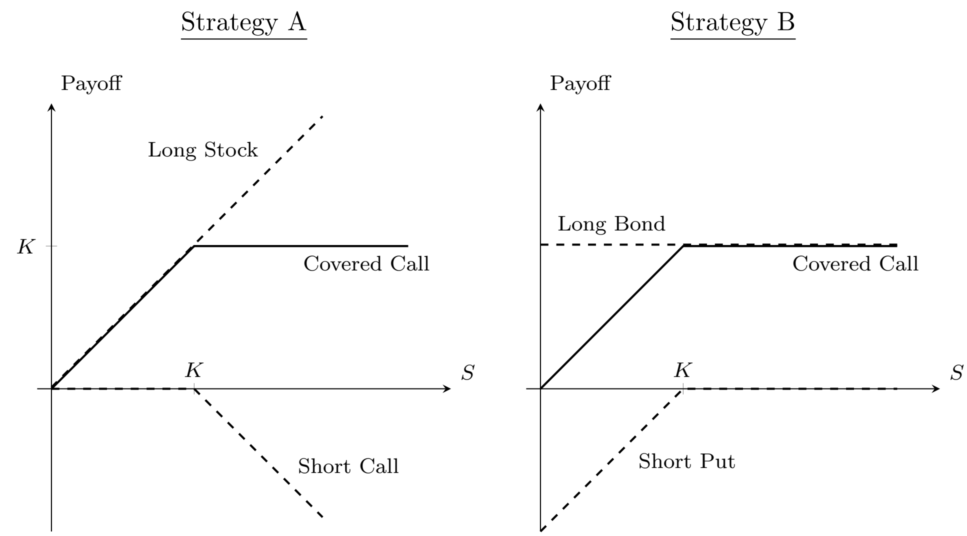


Figure 1: The figure shows that there are two equivalent ways to create a covered call strategy.

parity, we should have that a European put with the same strike and maturity as the call should cost:

$$\begin{aligned} P &= C - S + Ke^{-rT} \\ &= 13.30 - 110 + 110e^{-0.05\times 0.75} \\ &= 9.25 \end{aligned}$$

□

Example 2. What if in the previous example everything stays the same, but you find that the put trades for $9? Then we have an arbitrage opportunity. Let us consider both strategies discussed previously that we know have the same payoff.

Strategy A: Long stock and short call

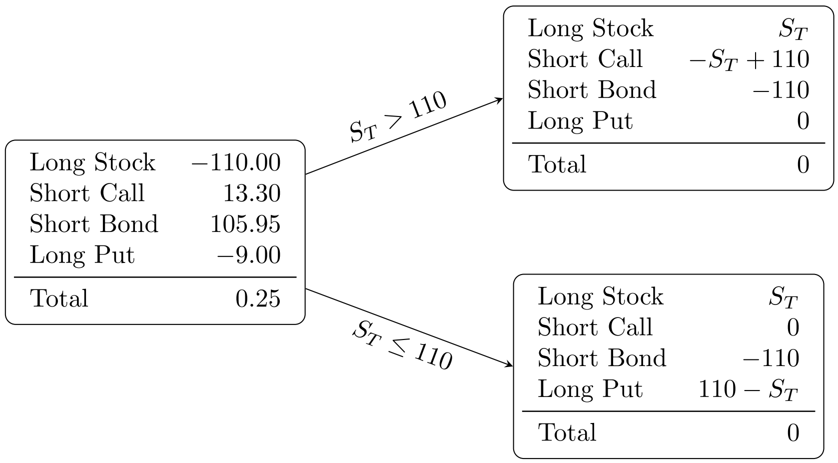
$$\text{Cost}_A = 110 - 13.30 = 96.70$$

Strategy B: Long bond and short put

$$\text{Cost}_B = 110e^{-0.05\times 0.75} - 9 = 96.95$$

Since $\text{Cost}_A < \text{Cost}_B$, we should buy A and sell B which generates an instant profit of

$0.25 per share. The diagram below shows that the strategy is indeed an arbitrage since the cash flows of the strategy are effectively zero at maturity but generates a positive cash flow today.



□

Put-Call Parity and Protective Puts

We can also express put-call parity in the following way:

$$S + P = Ke^{-rT} + C$$

The left hand-side of this expression is the cost of a covered put, i.e. long stock and long put. The right hand-side says that a protective put can also be built by buying a bond and a call.

The figure below shows the payoffs of a protective put built using each strategy.

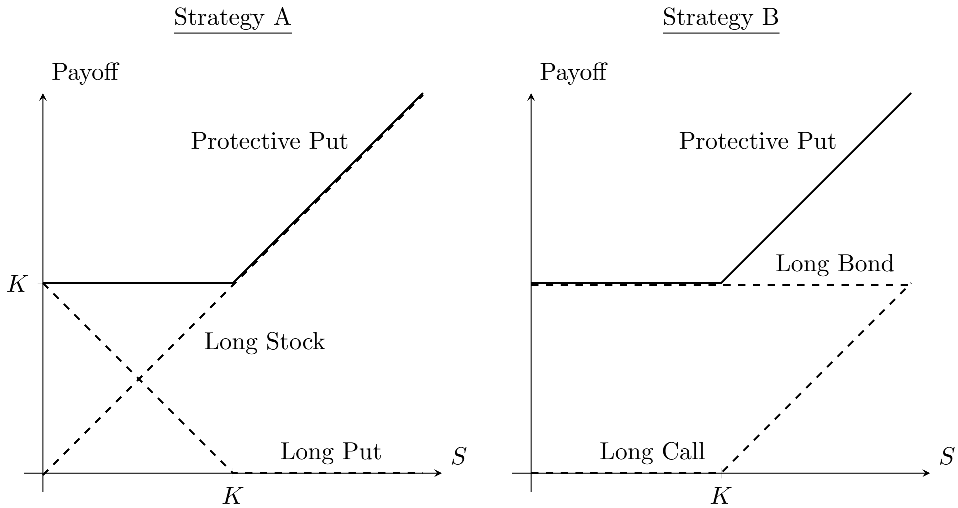


Figure 2: The figure shows that there are two equivalent ways to create a protective put strategy.

Put-Call Parity and Forward Contracts

We can express put-call parity in yet a different way:

$$C - P = S - Ke^{-rT}$$

The right hand-side of this expression is the cost of a forward contract with forward price K. The left hand-side says that a forward contract can be synthesized by buying a call and selling a put.

The figure below displays the payoff diagrams for a forward contract built using both strategies.

Bounds on European Call Options

Lower Bound on European Call Options

The price of a European call or put option must be positive, i.e. we must have that $C \geq 0$ and $P \geq 0$. [1]

[1] Note that if $T > 0$ then it must also be the case that $C > 0$ if $S > 0$, and $P > 0$ for any $S \geq 0$.

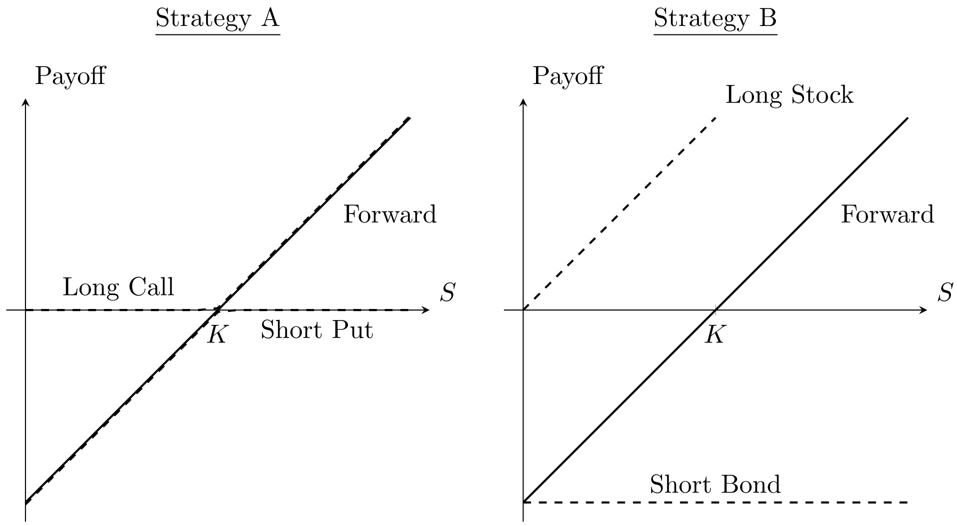


Figure 3: The figure shows that there are two equivalent ways to create the payoff of a forward contract.

If not, any trader would like to get as many contracts as possible. The worst-case scenario is that the options expire OTM, in which case the payoff is zero. Otherwise, the options expire ITM and the option trader gets a positive payoff. This would clearly be a nice arbitrage opportunity!

Furthermore, remember that we are analyzing an underlying asset that does not pay dividends. Put-call parity and the fact that $P \geq 0$ implies that:

$$C = P + S - Ke^{-rT} \geq S - Ke^{-rT}$$

Given that we also have $C \geq 0$, it must the case that:

$$C \geq \max(S - Ke^{-rT}, 0).$$

Example 3. Take the data from the previous example where $S = 110$ and $r = 5\%$ per year with continuous compounding. Consider a call option with strike \$110 and maturity 9 months. It must be the case that:

$$C \geq \max(110 - 110e^{-0.05\times0.75}, 0) = \max(4.05, 0) = 4.05$$

Hence, no matter how low the volatility is on this European call option, its premium must be higher than \$4.05. Otherwise it would be a violation of put-call parity. □

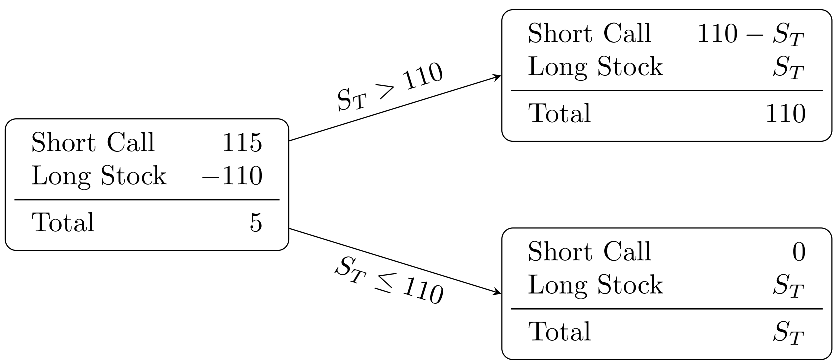
Upper Bound on European Call Options

On the other hand, the price of a European call on a non-dividend paying asset must cost less than the stock itself:

$$C \leq S$$

If not, it would make sense to write a call and use part of the proceeds to buy a share of stock.

Example 4. Assume that $S = 110$, $K = 110$, $T = 0.75$ years and $C = 115$.



In this case, there is indeed an arbitrage since the initial cash flow is positive and the final cash flow is also positive regardless of the value of the stock. In other words, this strategy makes money for free at $T = 0$ and keeps making money at $T = 0.75$! □

Feasible Prices for European Call Options

The graph describes the region of feasible prices for European call options written on a non-dividend paying asset when the risk-free rate is *positive*.

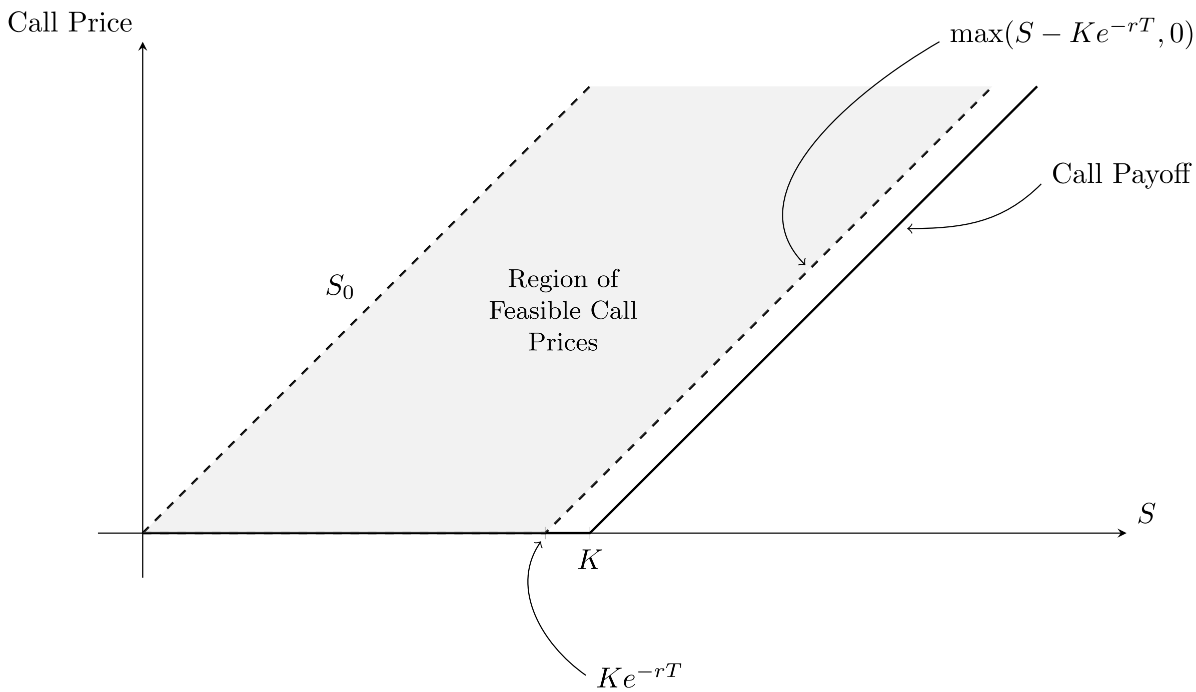


Figure 4: Region of feasible prices for European call options written on a non-dividend paying asset.

Therefore, when the risk-free rate is positive, the lower bound guarantees that the time value of a European call written on a non-dividend paying asset is always positive when the option is in-the-money.

The figure below displays the hypothetical price of a European call option for different values of the stock price S where $r = 5\%$, $T = 2$ years and $K = \$100$. It also shows the call option payoff given by $\max(S - K, 0)$ and the lower bound for a European call given by $\max(S - Ke^{-rT}, 0)$.

We can see that the time value of the call is positive when the option is in-the-money. Since the premium of an otherwise equivalent American call cannot be less than the premium of the European call, the graph also shows that the time value of the American call is positive when the option is in-the-money, which is the only case when it might be optimal to exercise early.

If the time value of an American call is positive, it means that early-exercising the option when it is in-the-money destroys value. Indeed, if the American call is exercised early it

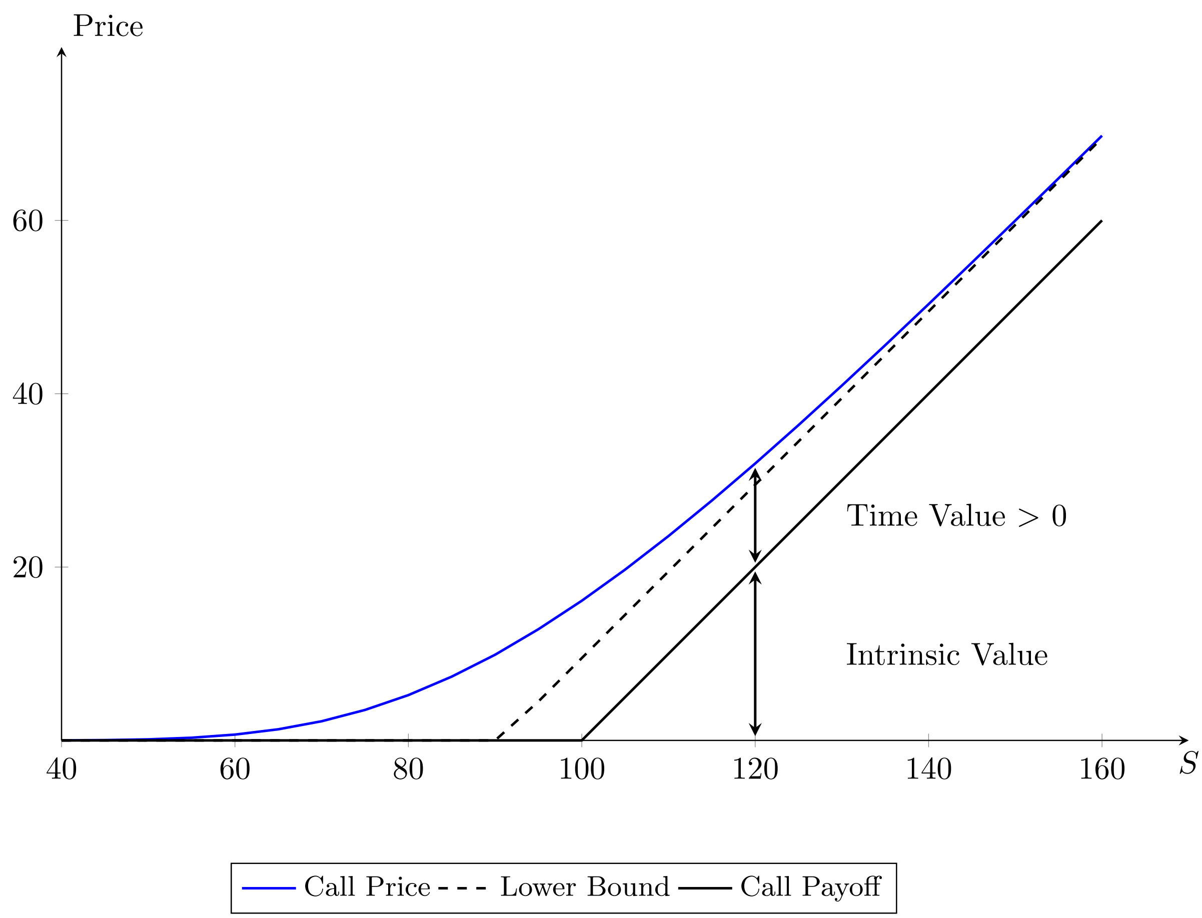


Figure 5: Hypothetical prices for a call option written on a noo-dividend paying asset.

would only pay its intrinsic value, which is lower than the option's premium. Therefore, it makes sense to keep the option alive and sell it in case the buyer would like to close the position.

Bounds on European Put Options

Put-call parity and the fact that $P \geq 0$ implies that:

$$P = C - S + Ke^{-rT} \geq -S + Ke^{-rT}$$

Given that we also have $C \geq 0$, it must the case that:

$$P \geq \max(Ke^{-rT} - S, 0)$$

Also, the maximum amount of money one can lose by writing a European put is K, which in present value terms is equal to Ke^{-rT}, implying that:

$$P \leq Ke^{-rT}$$

Feasible Prices for European Put Options

The graph describes the region of feasible prices for European put options written on a non-dividend paying asset when the risk-free rate is positive.

The figure below displays the hypothetical price of a put option for different values of the stock price S where $r = 5\%$, $T = 2$ years and $K = \$100$. It also shows the put option payoff given by $\max(K - S, 0)$ and the lower bound for a European put given by $\max(Ke^{-rT} - S, 0)$.

We can see that unlike the European call, the time value of the put is negative for values of the stock that are low enough. Indeed, in the extreme case that $S = 0$, the holder of the European put will receive a certain payoff of K at maturity. The value of that certain payoff today is $Ke^{-rT} < K$. Intuitively, the holder of the European put would like in this

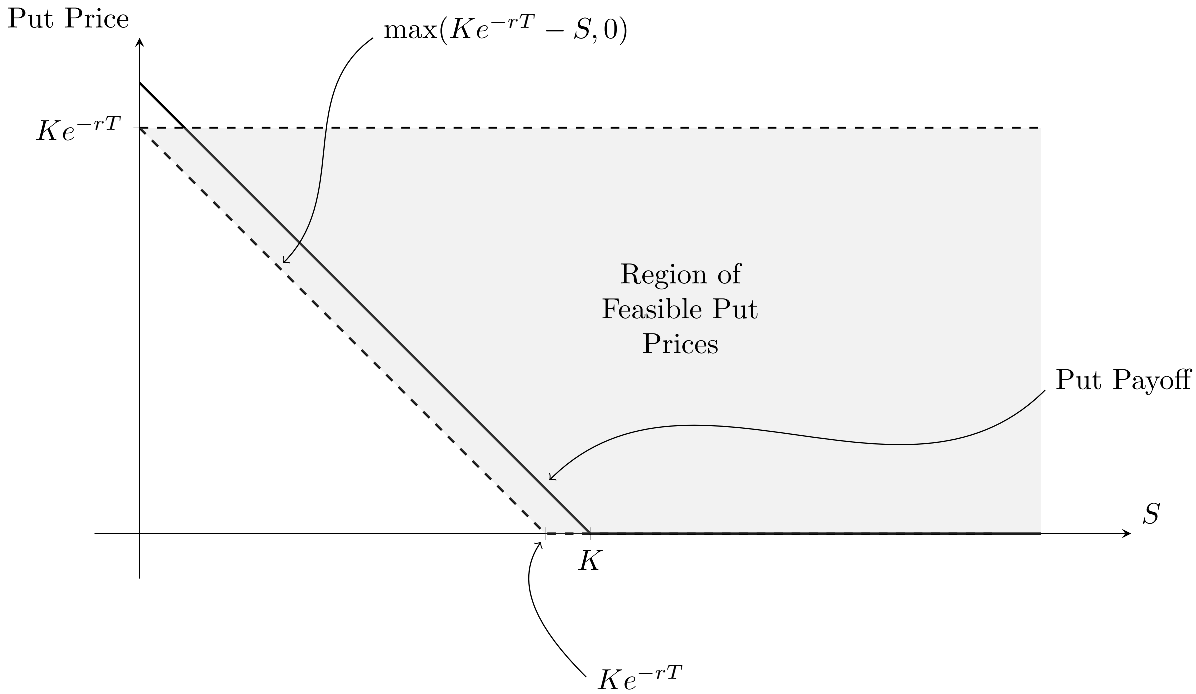


Figure 6: Region of feasible prices for European put options written on a non-dividend paying asset.

case to exercise immediately. But because the option is European she must wait until maturity to do so.

Impact of Negative Interest Rates

Even though traditionally the analysis so far assumed that $r > 0$, in the recent past interest rates in many countries have been negative, even for long maturities.

For a European call option, when $r < 0$ its lower bound is less than its intrinsic value, i.e., for a sufficiently high S the option will have negative time value and it might be optimal to early exercise an American call option.

The figure below displays the hypothetical price of a call option for different values of the stock price S where $r = -5\%$, $T = 2$ years and $K = \$100$. It also shows the call option payoff given by max$(S - K, 0)$ and the lower bound for a European call given by max$(S - Ke^{-rT}, 0)$.

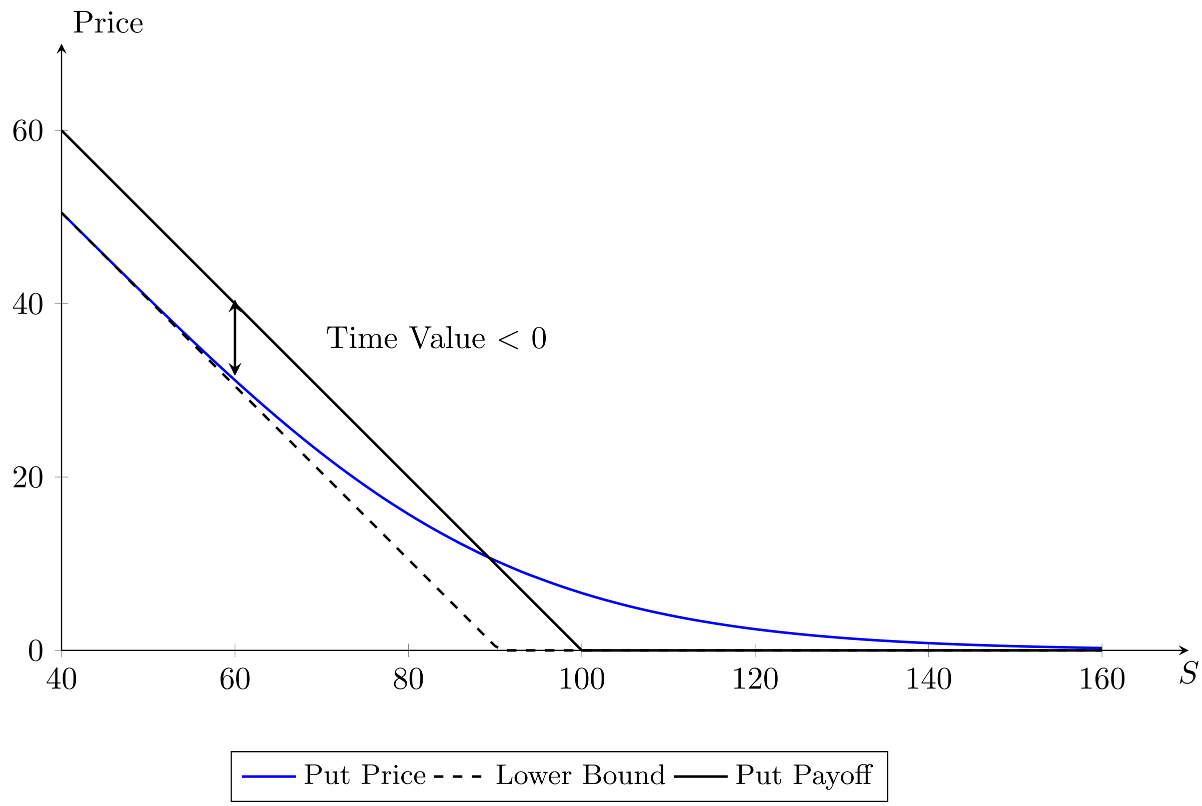


Figure 7: Hypothetical prices for a put option written on a noo-dividend paying asset.

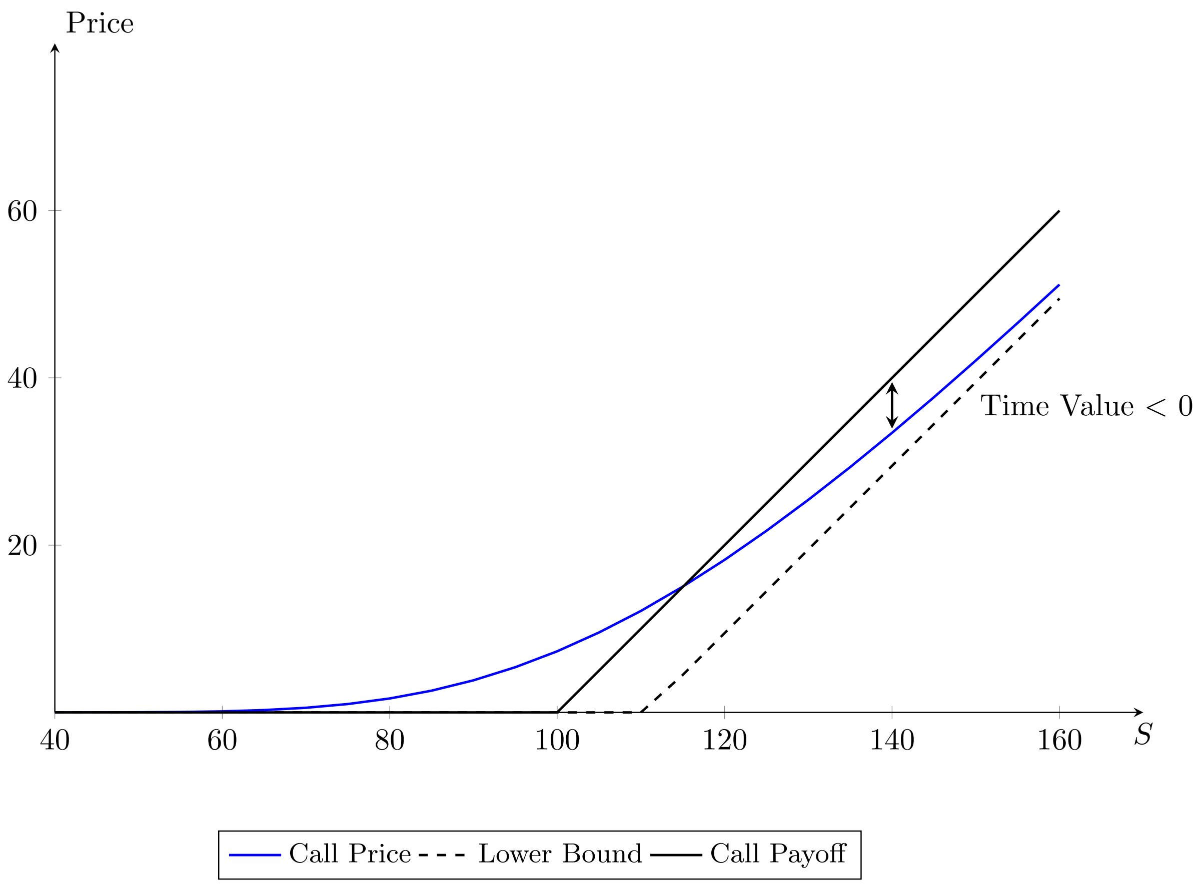


Figure 8: Hypothetical prices for a call option written on a noo-dividend paying asset when the interest rate is negative.

For put options, negative interest rates means that a European put option always has positive time value, i.e., it is not optimal to exercise early an American put option. Standard results that are usually taught in derivative courses get reversed!

The figure below displays the hypothetical price of a European put option for different values of the stock price S where $r = -5\%$, $T = 2$ years and $K = \$100$. It also shows the put option payoff given by $\max(K - S, 0)$ and the lower bound for a European put given by $\max(Ke^{-rT} - S, 0)$.

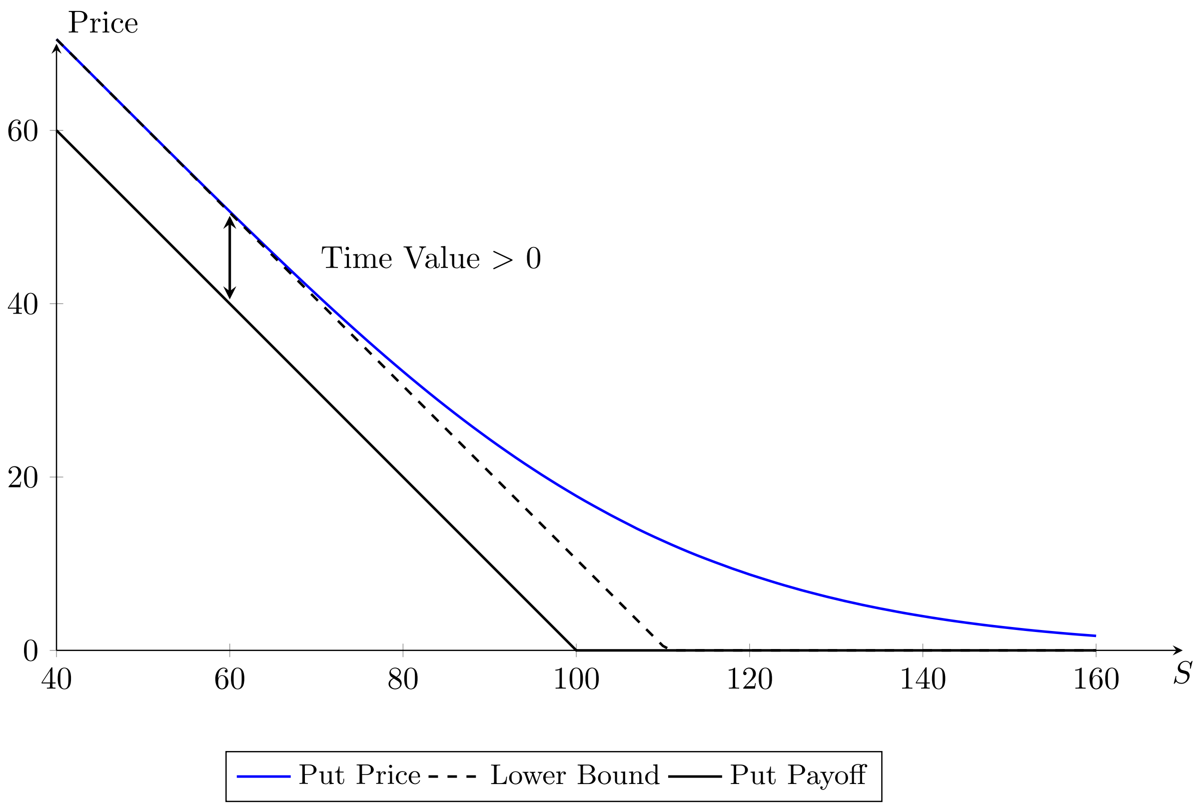


Figure 9: Hypothetical prices for a put option written on a noo-dividend paying asset when the interest rate is negative.

Factors Affecting European Option Prices

The analysis so far gives us some precise bounds on European option prices given the current stock and strike price, time-to-maturity and the risk-free rate. In this section we want to think about the effect that each one of these variables has on the price of a European call or put option written on a non-dividend paying asset.[2]

Variable	European Call	European Put
Current stock price	$+$	$-$
Strike price	$-$	$+$
Time-to-expiration	?	?
Volatility	$+$	$+$
Risk-free rate	$+$	$-$

For European options, the *current stock price* has an unambiguous effect on the option premium. Indeed, if the stock price suddenly goes up, a call option is more ITM whereas a put option becomes more OTM. The *strike price* has an opposite effect. A call option with a lower strike allows the buyer to purchase the asset for less at maturity and hence the option should cost more. The opposite is true for a put since a higher strike price allows the buyer of the put to sell the asset for more at expiration.

The effect of *time-to-expiration* on the option premium has an ambiguous effect for European options, even when the underlying asset does not pay dividends. The lower bound for a European call on a non-dividend paying asset is above intrinsic value if the interest rate is positive. Its time-value is then positive regardless of the value of the stock, and always decreases as we approach maturity.

On the other hand, the lower bound for a European call on a non-dividend paying asset is below intrinsic value if the risk-free rate is negative. The time-value of deep ITM calls is negative and therefore increases as time-to-maturity decreases. The time-value of OTM and some ITM call options is positive and decreases as we approach maturity. A reverse analysis applies to European put options.

The *volatility* of the stock price returns has a positive effect for both European calls and puts, and is one of the most important factors driving options prices. The fact that the option payoff is capped below at zero makes the option more attractive when volatility

[2] Note the analysis considers that when we change, say, the stock price, all other variables are kept constant. This is similar to when in calculus you take a partial derivative.

is high, thus increasing its price. Practitioners usually refer to the sensitivity of option prices with respect to volatility as *vega*.

Finally, the *risk-free rate* has a positive effect on European call options but a negative effect on European put options. Indeed, the reason why call options are so risky is because they behave *as if* they are levered positions on the stock. In other words, a European call option looks like a position where you have massively borrowed in order to buy stocks. Hence the perfect positive correlation with the stock and higher risk. Therefore, call prices are positively related to increases in the risk-free rate.

On the contrary, a put option can be thought as a positive where you short-sell the stock in order to invest in risk-free bonds. This means that the sensitivity of put option prices to increases in interest rates is negative. Market practitioners refer to the sensitivity of option prices with respect to interest rates as *rho*.

Summary

We summarize the results of options bounds for European call and put options on a non-dividend paying asset below.

Bounds on European Options on Non-Dividend Paying Assets

We have the following bounds for European call and put options written on a non-dividend paying asset:

$$\max(0, S - Ke^{-rT}) \leq C \leq S$$
$$\max(0, Ke^{-rT} - S) \leq P \leq Ke^{-rT}$$

Example 5. The risk-free rate is 10% per year with continuous compounding. Furthermore, assume that $S = 50$, $K = 45$, and $T = 1.20$. Let us compute the bounds for European call and put options. First, for the put option we have that:

$$0 = \max(45e^{-0.10\times1.20} - 50, 0) \leq P \leq 45e^{-0.10\times1.20} = 39.91$$

Second, for the call option:

$$10.09 = \max(50 - 45e^{-0.10\times 1.20}, 0) \leq C \leq 50$$

In this case the lower bound for the put is zero but the lower bound for the call is positive.

□

Practice Problems

Solutions to all problems can be found at lorenzonaranjo.com/fin451.

Problem 1. What is a lower bound for the price of a four-month European call option on a non-dividend-paying stock when the stock price is $28, the strike price is $25, and the risk-free interest rate is 8% per year?

Problem 2. What is a lower bound for the price of a one-month European put option on a non-dividend paying stock when the stock price is $12, the strike price is $15, and the risk-free interest rate is 6% per year?

Problem 3. What is a lower bound for the price of a six-month European call option on a non-dividend-paying stock when the stock price is $80, the strike price is $75, and the risk-free interest rate is 10% per year?

Problem 4. What is a lower bound for the price of a two-month European put option on a non-dividend paying stock when the stock price is $58, the strike price is $65, and the risk-free interest rate is 5% per year?

Problem 5. A one-month European put option on a non-dividend-paying stock is currently selling for $2.50. The stock price is $47, the strike price is $50, and the risk-free interest rate is 6% per year. What opportunities are there for an arbitrageur?

Problem 6. A non-dividend paying stock trades for \$120. The risk-free rate is 5% per year with continuous compounding. European call and put options with strike \$120 and maturity 9 months trade for \$8 and \$4 per share, respectively. Is there an arbitrage opportunity? If so, how an arbitrageur would make a profit?

Problem 7. A non-dividend paying stock trades for \$200. The risk-free rate is 8% per year with continuous compounding. A European call option with strike \$200 and maturity 1 year trades for \$14. This means that:

a. The put price is -\$1.38.
b. You should sell the call, buy the stock and borrow \$184.62 at the risk-free rate for 1 year.
c. You should buy the call, sell the stock and invest \$184.62 at the risk-free rate for 1 year.
d. The put price is \$1.38.

Problem 8. The price of a non-dividend paying stock is \$100. The risk-free rate is 8% with continuous compounding. If C denotes the premium of a call option with strike \$100 and maturity 6 months. What are the tightest bounds for the call?

Problem 9. The price of a non-dividend paying stock is \$300. The risk-free rate is 5% per year with continuous compounding. Consider a European put option with strike price \$320 and maturity 1 year. What is the lowest price at which the put can sell?

Problem 10. Consider a non-dividend paying asset. There are European call and put options written on this asset and available for trade. If the risk-free rate is negative, then:

a. It might be optimal to exercise an American put option written on an asset that does not pay dividends.
b. The time-value of a European call might be negative for stock prices that are high enough.
c. It is never optimal to exercise an American call option written on an asset that does not pay dividends.
d. The time-value of a European put might be negative for stock prices that are high enough.