Prof. Lorenzo Naranjo Spring 2025

Forward Contracts

Definitions

A long forward requires the buyer to purchase the asset at expiration for the forward price prevailing when the contract started, which we denote by K. If the price of the asset at maturity is S, then the payoff of the long position in the forward contract is $S - K$.

Indeed, suppose the asset price at maturity is greater than the forward price. In that case, we purchase the asset by K and immediately sell it by S, generating a profit of $S - K$. If, on the other hand, $S < K$, then the payoff is negative since under the terms of the contract we are required to purchase the asset for K which we can only sell at a lower price S.

Figure 1 plots the payoff of a forward contract when the forward delivery price is K, as a function of the spot price S. As can be seen from the picture, the payoff of a long forward increases in S and cuts the x-axis at the forward delivery price of K.

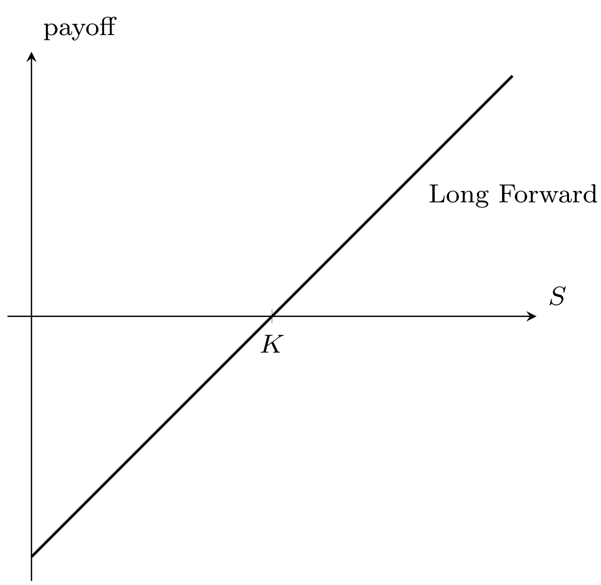

Figure 1: The payoff function of a long forward.

Figure 2 shows the payoff of a short forward position, given by $K - S$. The short forward payoff is then the mirror image with respect to the x-axis of the long forward position. By fixing the price at K, the seller is happy when prices go down but is unhappy when prices increase.

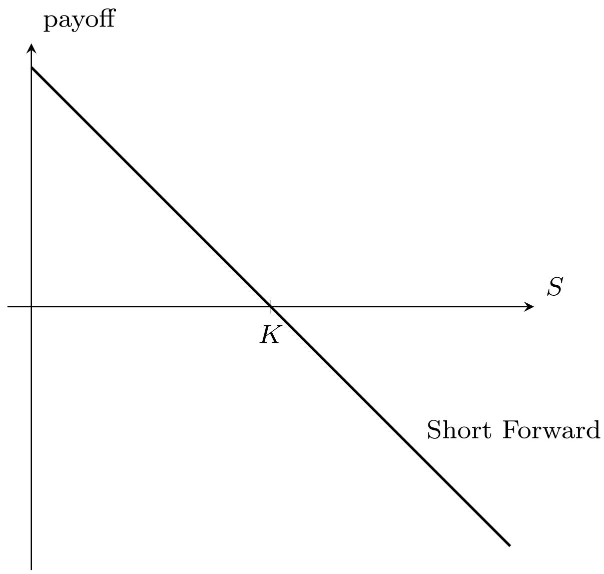

Figure 2: The payoff function of a short forward.

Example 1 (Payoff from buying currency forward)**.** On May 24, 2010, the treasurer of a corporation entered into a long forward contract to buy £1 million in six months at an exchange rate of $1.4422 per pound sterling. The agreement binds the corporation to pay $1,442,200 for £1 million on November 24, 2010. What are the possible outcomes?

We can compute the contract payoff for different exchange rate values six months from now, giving us an idea of the possible outcomes depending on how the exchange rate is in six months.

Exchange Rate	1.2000	1.3000	1.4000	1.5000	1.6000
Payoff	-242,200	-142,200	-42,200	57,800	157,800

The figure below plots the payoff of this long forward contract as a function of the currency exchange rate at maturity.

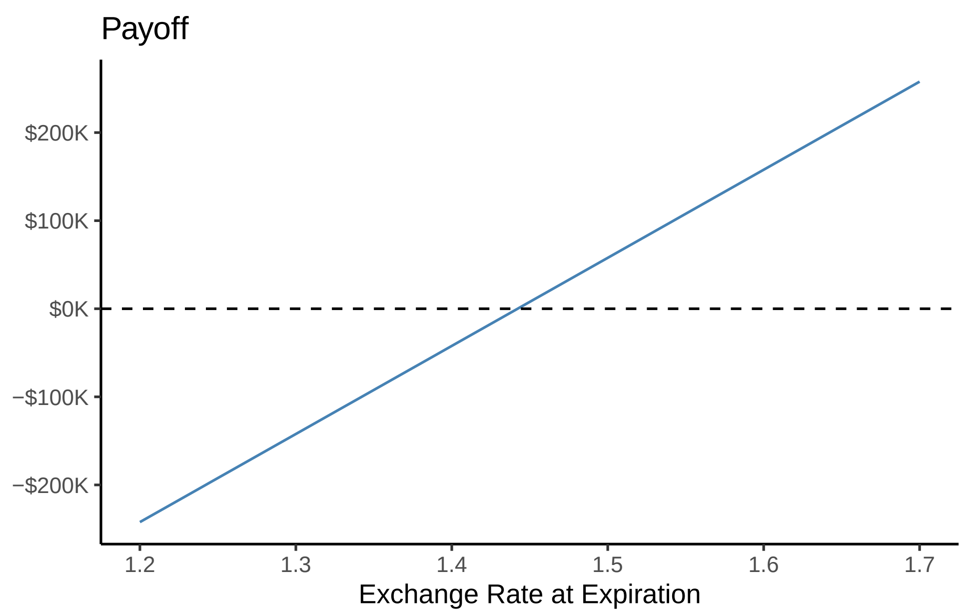

Figure 3: The payoff of a long forward contract as a function of the currency exchange rate at maturity.

Forward Prices

We now focus on understanding what should determine the forward price in equilibrium. The *forward price* is the delivery price applicable to a forward contract with zero value. The party that has agreed to buy at the forward price has a *long position* whereas the party that has decided to sell at that price has a *short position*.

We compute the forward price by assuming that it is hard to make easy money without taking any risk in financial markets. We call this notion the *principle of no-arbitrage*.

An *arbitrage* is a transaction that would cost nothing or even provide income today and even more income in the future. Even if we do not get anything later, getting money for free today is a pretty good deal! The deal is so good that it would only last for a while. An arbitrage opportunity is inconsistent with an economic equilibrium, but the idea is more general. The no-arbitrage principle does not depend on agents' preferences, and as such, it holds in any economic system. All we need is to assume that it is hard to make easy money.

Non-Dividend Paying Assets

Forward Price of a Non-Dividend Paying Asset

If the spot price of a non-dividend paying asset is S, then the forward price for a contract deliverable in T years is given by

$$F = Se^{rT}, \tag{1}$$

where r is the continuously compounded risk-free rate corresponding to the forward contract expiration.

We start by computing the *no-arbitrage* forward price of a non-dividend paying asset. How do we know that this price is correct? We will see shortly that if the forward price differs from the one given in (1), then there is an arbitrage opportunity that specialized traders would have no problem exploiting at a massive scale. But first, let us see how to use this expression.

Example 2. Suppose that the current price of a non-dividend-paying stock is \$40, the 3-month forward price is \$43, and the 3-month US\$ interest rate is 5% per year with continuous compounding. The no-arbitrage forward price is:

$$F = 40e^{0.05\times 0.25} = 40.50.$$

The following example shows what would happen if the forward price of the contract described in Example 2 was higher than \$40.50. When an asset is trading for more than its fair price, it makes sense to sell it.

Example 3. Suppose that the current price of a non-dividend-paying stock is \$40, the 3-month forward price is \$43, and the 3-month US\$ interest rate is 5% per year with continuous compounding. Is there an arbitrage opportunity?

The table below shows the cash flows that an investor would get today and in three months if she sells one forward contract, borrows the present value of \$43 to be paid in three months, and buys the stock. Therefore, borrowing means a positive cash flow today and a negative cash flow in the future, whereas buying a stock means a negative cash flow today and a positive cash in the future when we sell the stock.

	$T = 0$	$T = 3/12$
Short forward	0.00	$43 - S_T$
Borrow	42.47	-43
Long stock	-40.00	S_T
Total	2.47	0

The table shows that the *cost* of this strategy is *negative* \$2.47, i.e., you would make money today by engaging in these transactions. Furthermore, today's positive cash flow has zero risk since we can see that the cash flow in three months is zero. Since by selling the forward for \$43, we can make an arbitrage profit, we conclude that the forward price, in this case, is too high.

We now focus on the case when the observed forward price is less than its fair price. If this were the case, buying the forward and short-selling the stock would make sense.

Example 4. Suppose that the current price of non-dividend paying stock is \$40, the 3-month forward price is US\$39, and the 3-month US\$ interest rate is 5% per year with continuous compounding. Is there an arbitrage opportunity?

The table below shows an investor's cash flows today and in three months if she buys one forward contract, invests the present value of \$39 to get that amount in three months, and sells the stock. Notice that she must not own the stock to sell it short.

Again, the cost of this strategy is negative as it generates money today. As before, the trader faces no risk in three months. There is an arbitrage opportunity because the forward price is too low.

	$T = 0$	$T = 3/12$
Long forward	0.00	$S_T - 39$
Invest	-38.52	39
Short stock	40.00	$-S_T$
Total	1.48	0

As we mentioned before, the no-arbitrage forward price defined in (1) works for any non-dividend paying asset, even a precious metal like gold, as long as we understand that S represents the spot price and not necessarily the cash price of the commodity.

Example 5. Suppose the gold price is currently $1,870.60 per ounce, and consider a forward contract on gold expiring in one year. Assume that the cost of storing gold is negligible and no additional benefits are accruing from owning gold. The risk-free rate is 5% per year with continuous compounding. Then, the no-arbitrage forward price of gold is

$$F = 1{,}870.60e^{0.05} = 1{,}966.51.$$

Note that we do not need to know what gold prices will do in the future to fix the price today, at which we can buy gold in one year.

Assets Paying Cash Dividends

Forward Price of an Asset Paying Cash Dividends

The forward price of an asset paying cash dividends is given by

$$F = (S - D)e^{rT}, \tag{2}$$

where S is the current asset price, T is the maturity of the forward, D denotes the present value of the dividends or net income earned during the contract's life, and r denotes the continuously compounded interest rate.

Many assets will pay dividends or income during the contract's life. Note that there might be some non-negligible storage costs for some commodities, which implies that the total income you derive from owning the asset is net of any expenses of having the commodity in storage. The net dividend might become negative if the storage costs outweigh the benefits of owning the commodity. Holding a long forward contract might be better than the commodity itself.

We will denote by D the present value of the dividends or net income accruing to the owner of the asset or physical commodity but not to the buyer of a forward contract during the *life of the contract*.

For example, think about a stock. In this case, the income is the quarterly dividends you receive if you own the stock. If you were instead holding a long position on a forward contract written on the stock, you would not be entitled to any dividend payments during the contract's life. The forward price needs to be adjusted accordingly.

We note that $D = 0$ when $T = 0$ since the asset pays no dividends if the contract expires immediately. In this case, the forward price is indeed equal to the asset price when $T \to 0$. Certainly, it would have been more accurate to denote D by $D(T)$, but this would have made the notation cumbersome.

Example 6. Consider a stock that currently trades at $50. Analysts expect the stock to pay dividends of $1.15 and $1.20 in two and five months, respectively. If the risk-free rate is 5% per year with continuous compounding, let us compute the forward price six months from now.

For this, we need to compute the present value of the dividends. Even though, in practice, dividend payments are uncertain, we could assume they are known for short-term maturities, and hence, we can discount them using the risk-free rate. We have that

$$D = 1.15e^{-0.05\times 2/12} + 1.20e^{-0.05\times 5/12} = 2.32.$$

Therefore,

$$F = (50 - 2.32)e^{0.05\times 6/12} = 48.89.$$

The forward price we computed in Example 6 is the price that prevents any arbitrage opportunities. If the forward price differed, an arbitrageur could engage in the following strategy and profit arbitrarily.

Example 7. Consider the same stock described in Example 6. What would an arbitrageur do if the six-month forward price was $50.20?

We know that the no-arbitrage forward price is $48.89. If the actual forward price is $50.20, it means that the six-month forward is too expensive, so we should sell it. We hedge the transaction by buying one share of the stock at $50. The timeline below shows that an arbitrageur selling the six-month stock forward and buying the stock at time 0 is entitled to dividends in two and five months, plus the stock itself that somebody could sell at the six-month forward price prevailing today.

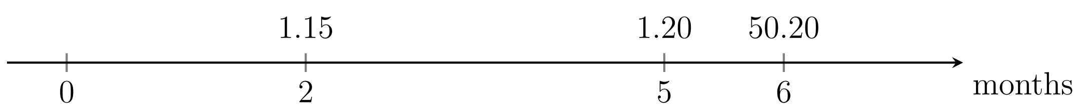

Figure 4: The figure shows the cash flows paid by the stock.

All this suggests that the arbitrageur could take the following loans to be repaid in full using the dividend payments and the sale of the stock in six months:

- *Loan 1*: Borrowing $1.15e^{-0.05\times 2/12} = \$1.14$ today and repaying $1.15 in two months.
- *Loan 2*: Borrowing $1.20e^{-0.05\times 5/12} = \$1.18$ today and repaying $1.20 in five months.
- *Loan 3*: Borrowing $50.20e^{-0.05\times 6/12} = \$48.96$ today and repaying $50.20 in six months.

The table below describes the cash flows received by the arbitrageur when engaging in this transaction. A positive cash flow means the arbitrageur receives money, whereas a negative cash flow means the opposite.

	$T = 0$	$T = 2/12$	$T = 5/12$	$T = 6/12$
Short forward	0.00			$50.20 - S_T$
Loan 1	1.14	-1.15		
Loan 2	1.18		-1.20	
Loan 3	48.96			-50.20
Long stock	-50.00	1.15	1.20	S_T
Total	1.28	0	0	0

This strategy generates a sure profit of $1.28 per share of the stock and has *no risk*. An arbitrageur could sell 100 million forwards and hedge accordingly to generate an instantaneous risk-free profit of $128 million.

Assets Paying a Dividend Yield

Forward Price of an Asset Paying a Dividend Yield

The forward price of a dividend-yield paying asset is given by

$$F = Se^{(r-\delta)T}$$

where S is the current asset price, T is the maturity of the forward, δ is the continuous dividend or convenience yield, and r denotes the continuously compounded interest rate.

Many assets pay dividends continuously, like a foreign currency. Some other assets behave *as if* they pay a continuous dividend like a stock index such as the S&P 500. In these cases, it is convenient to consider dividends as a percentage yield paid over time. Given the convenience of this approach, some practitioners also use it to model individual stocks, even though the dividends are paid quarterly in these cases. In what follows, we will denote the continuously compounded dividend yield as δ.

The asset S then pays every instant t over a time-period Δt a dividend of $\delta \Delta t$ units of the asset. Therefore, the dividend yield represents the units of the asset growing over time at the rate of δ. Thus, if we start with one share at 0 after T years, we will have $e^{\delta T}$ shares.

In the next section, we analyze the market for foreign currencies, which is a typical example of an asset paying a dividend yield.

Currency Forwards

Foreign Currencies

The exchange rate between two currencies is the number of domestic currency units per unit of foreign currency. We must be careful, though, since the street market convention for the EUR/USD exchange rate implies that the *quote* currency is the US dollar (USD), and the *base* currency is the Euro (EUR). For example, the direct quotation of the EUR/USD could be $1.08/€ and represents the price in USD of 1 EUR. Note that you could always define it the other way around (indirect quotes), which practitioners do for many currency pairs.

The market convention of calling this exchange rate EUR/USD might induce errors. Even though we write EUR/USD, EUR-USD, or EURUSD, it represents the number of USD per EUR, i.e., $\$1.08 \Leftrightarrow$ €1. Be careful, though, as in some textbooks, you might find it the other way around.

Example 8. If the EUR/USD exchange rate is 1.08, for a US investor, 1 Euro is worth $1.08, but in Europe, how many Euros is worth $1?

$$\$1 = \frac{1}{1.08} = €0.93/\$.$$

Therefore, to buy $1, you must pay €0.93.

Some currency pairs, such as EUR/USD or GBP/USD, use the USD as the quote currency. However, most currency pairs are expressed using the dollar as the base currency, i.e., USD/JPY, USD/CNY, USD/CLP, etc.

Forward Price of a Foreign Currency

Currency forwards are contracts written on a foreign currency and trade over-the-counter (OTC). The FX forward market is one of the largest in the world. As of 2019, the Bank for International Settlements (BIS) reports that the daily turnover of currency forwards is approximately $1 trillion.

When pricing currency forwards, the dividend yield represents the interest rate you would earn if you had a certain amount of foreign currency in a deposit account. The asset's spot price is the foreign currency's price in US dollars (USD).

To compute the currency forward price, we need to know the current spot currency price, as well as the interest rates of the foreign and domestic currency. Currency forward prices are usually expressed as forward points, 10,000 times the difference between the forward and the spot price.

Example 9. The current GBP/USD exchange rate is 1.30. The interest rates in USD and GBP are 1% and 3% per year with continuous compounding, respectively. The 9-month GBP/USD forward price is then,

$$F = 1.30e^{(0.01-0.03)\times 9/12} = 1.2806,$$

or $10{,}000 \times (1.2806 - 1.3000) = -193.5$ forward points.

Valuing an Existing Forward Position

A forward contract is worth zero at inception. Later, it may have a positive or negative value since the underlying asset might increase or decrease in value, and the time-to-maturity of the contract decreases.

Suppose you bought the forward some time ago for K, and you would like to know how much that contract is worth today. If the current forward price is K, you could sell the forward today and completely hedge your future exposure. Indeed, in the past, you committed to purchasing the asset for K at maturity, but you just committed to selling it for

F on the same date. You have just locked in a certain cash flow of $F - K$ at time T, which in present value terms is worth today $(F - K)e^{-rT}$.

To value an existing short forward position entered some time ago at a forward price of K, you could buy a forward contract at F today, locking in a certain cash flow of $K - F$ at maturity. The value of the short forward contract is then $(K - F)e^{-rT}$.

Example 10. You entered into a short-forward contract some time ago on an asset that pays a dividend yield of 7% per year. The forward price at that time was $200. Today, the contract has six months until maturity, and the current forward price is $190. Also, the current risk-free rate is 5% per year with continuous compounding.

To compute the current value of the short forward position, we could imagine what would happen if we buy a forward today. That would lock in a sure cash flow in six months of $200 - 190 = \$10$, whose present value today is $10e^{-0.05\times 6/12} = \$9.75$, the value of the short forward contract.

Practice Problems

For solutions go to lorenzonaranjo.com/fin451.

Problem 1. A forward contract is defined as:

a. The right but not the obligation to purchase or sell an asset in the future at a given fixed price for a given quantity.
b. A commitment to purchase or sell an asset at a given fixed price for a given quantity.
c. An asset that pays a fixed amount at maturity.
d. A physical position in the underlying asset.

Problem 2. Suppose you enter into a 6-month forward contract on a non-dividend-paying stock when the stock price is $30, and the risk-free interest rate is 12% per year with continuous compounding. What is the forward price?

Problem 3. Imagine that you agree to purchase 1 million euros in 9 months from now at an exchange rate of \$1.12 per euro. What would be your payoff if the exchange rate in 9 months is \$1.02 per euro?

Problem 4. A 1-year long forward contract on a non-dividend-paying stock is entered into when the stock price is \$40, and the risk-free rate of interest is 10% per annum with continuous compounding.

a. What are the forward price and the initial value of the forward contract?
b. Six months later, the price of the stock is \$45, and the risk-free interest rate is still 10%. What are the forward price and the value of the forward contract?