Prof. Lorenzo Naranjo Spring 2025

The Greeks

Packages

A *package* is a portfolio of standard options. The main difference between a package and an option strategy is that the package is sold as a whole product, whereas an option strategy involves trading different options at the same time.

We have studied many option strategies such as bull spreads, bear spreads, straddles, strangles, butterflies, and condors. Some of these strategies could be sold as a package.

One popular package is a *range forward* contract. When applied to foreign currencies, a range forward has the effect of ensuring that the exchange rate paid or received lies within a certain range.

A long range forward involves selling a put with strike K_1 and buying a call with strike K_2 (with $K_1 < K_2$). This would be similar to a long forward position.

A short range forward involves buying a put with strike K_1 and selling a call with strike K_2. This would be similar to a short forward position. Normally the price of the put equals the price of the call so the contract has zero cost.

Variations of the Black & Scholes Framework

Gap Options

A gap option uses different strike prices to determine when and how much to pay. For example, a gap call pays $S_T - K_1$ when $S_T > K_2$, and zero otherwise. Therefore, K_2 determines if the option is ITM at expiration whereas K_1 determines how much the option pays if it is ITM. Therefore, a gap call is equivalent to a European call with strike if $K_1 = K_2$.

The price of the gap call can then be obtained by computing its expected risk-neutral payoff and discounting it at the risk-free rate, that is,

$$\begin{aligned}\text{Gap Call} &= \mathsf{E}\left((S_T - K_1)\mathbb{1}_{\{S_T>K_2\}}\right) e^{-rT} \\ &= \mathsf{E}\left(S_T\mathbb{1}_{\{S_T>K_2\}}\right) e^{-rT} - K_1 e^{-rT}\, \mathsf{E}\left(\mathbb{1}_{\{S_T>K_2\}}\right) \\ &= Se^{-\delta T}\,\Phi(d_1) - K_1 e^{-rT}\,\Phi(d_2)\end{aligned}$$

where

$$d_1 = \frac{\ln(S/K_2) + (r - \delta + 0.5\sigma^2)T}{\sigma\sqrt{T}}$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

Example 1. Suppose you want to price an option that pays $S_T - 180$ if $S_T > 220$ in 10 months. The option is written on a non-dividend paying asset whose current price is \$200 and that has a volatility of returns of 40%. The risk-free rate is 6% per year with continuous compounding.

The option in the example is a gap call option where $K_1 = 180$ and $K_2 = 220$. This implies,

$$d_1 = \frac{\ln(200/220) + (0.06 + 0.5(0.40)^2)(10/12)}{0.40\sqrt{10/12}} = 0.0585$$

$$d_2 = d_1 - 0.40\sqrt{10/12} = -0.3067$$

The gap call price is then given by,

$$\text{Gap Call} = 200\,\Phi(0.0585) - 180e^{-0.06\times 10/12}\,\Phi(-0.3067) = \$39.68$$

A gap put is similarly defined. It pays $K_1 - S_T$ when $S_T < K_2$, and zero otherwise. Its price is given by

$$\text{Gap Put} = K_1 e^{-rT}\,\Phi(-d_2) - Se^{-\delta T}\,\Phi(-d_1)$$

Example 2. Using the data in example Example 1, the price of a gap put is

$$\text{Gap Put} = 180e^{-0.06\times 10/12}\,\Phi(0.3067) - 200\,\Phi(-0.0585) = \$10.90$$

Binary Options

Binary options pay either a certain amount of cash or units of a risky asset if the underlying asset is above or below a certain strike price. For example, a *cash-or-nothing* call could pay Q if the stock price is above the strike price K or nothing otherwise. An *asset-or-nothing* call could pay 1 unit of the stock if the stock price is above the strike K or nothing otherwise.

Interestingly, a European call option can be seen as a long *asset-or-nothing* call and a short *cash-or-nothing* call. Indeed, a European call pays $S-K$ if $S>K$ at maturity, which can be seen as receiving S and paying K whenever $S>K$, implying that

$$C = \underbrace{Se^{-\delta T}\,\Phi(d_1)}_{\text{asset-or-nothing call}} - \underbrace{Ke^{-rT}\,\Phi(d_2)}_{\text{cash-or-nothing call}}$$

Similarly, a European put can be seen as a long *cash-or-nothing* put and a short *asset-or-nothing* put:

$$P = \underbrace{Ke^{-rT}\,\Phi(-d_2)}_{\text{cash-or-nothing put}} - \underbrace{Se^{-\delta T}\,\Phi(-d_1)}_{\text{asset-or-nothing put}}$$

The following table shows the prices of *asset-or-nothing* options that pay 1 share of the stock, and *cash-or-nothing* options that pay Q, whenever the option is ITM at maturity.

Type	Call	Put
Asset-or-nothing	$Se^{-\delta T}\,\Phi(d_1)$	$Se^{-\delta T}\,\Phi(-d_1)$
Cash-or-nothing	$Qe^{-rT}\,\Phi(d_2)$	$Qe^{-rT}\,\Phi(-d_2)$

Example 3. Consider a stock that pays a dividend yield of 3% and that has a volatility of returns of 35%. The stock price is \$150 and the risk-free rate is 8%.

First, let's price an *asset-or-nothing* call that pays 1 share of the stock if the stock price in 6 months is above \$150. We can then compute

$$d_1 = \frac{\ln(150/150) + (0.08 - 0.03 + 0.5 \times 0.35^2) \times 0.5}{0.35\sqrt{0.5}} = 0.2248$$

The costs of the asset-or-nothing call is $150e^{-0.03\times 0.5}\,\Phi(0.2248) = \$87.02$. The price of an otherwise equivalent asset-or-nothing put is $150e^{-0.03\times 0.5}\,\Phi(-0.2248) = \$60.74$.

Example 4. Using the data in Example 3, consider a *cash-or-nothing* call that pays \$100 if the stock price in 6 months is above \$150. We can compute

$$d_2 = d_1 - 0.35\sqrt{0.5} = -0.0227$$

The cost of the cash-or-nothing call is then $100e^{-0.08\times 0.5}\,\Phi(-0.0227) = \$47.17$. The cost of an otherwise equivalent asset-or-nothing put is $100e^{-0.08\times 0.5}\,\Phi(0.0227) = \$48.91$.

Forward Start Options

A *forward start option* is a European call or put that starts at a future time τ and expires at time $T > \tau$. They are typically implicit in employee stock option plans, and are often structured so that strike price equals asset price at time τ, that is, $K = S_\tau$.

Therefore, the value of a call or put option at time τ is:

$$C_\tau = S_\tau e^{-\delta(T-\tau)}\,\Phi(d_1) - S_\tau e^{-r(T-\tau)}\,\Phi(d_2)$$
$$P_\tau = S_\tau e^{-r(T-\tau)}\,\Phi(-d_2) - S_\tau e^{-\delta(T-\tau)}\,\Phi(-d_1)$$

where $d_1 = \dfrac{(r-\delta+0.5\sigma^2)(T-\tau)}{\sigma\sqrt{T-\tau}}$ and $d_2 = d_1 - \sigma\sqrt{T-\tau}$.

The price of the option today is then $V = \mathrm{E}(V_\tau)e^{-r\tau}$ where the expectation is taken of course with respect the risk-neutral measure.

Noting that the futures price of a contract expiring at time τ is given by $f = \mathrm{E}(S_\tau) = Se^{(r-\delta)\tau}$, we have that $\mathrm{E}(S_\tau)e^{-r\tau} = Se^{-\delta\tau}$. Therefore:

$$C = \left(Se^{-\delta(T-\tau)}\,\Phi(d_1) - Se^{-r(T-\tau)}\,\Phi(d_2)\right)e^{-\delta\tau}$$
$$P = \left(Se^{-r(T-\tau)}\,\Phi(-d_2) - Se^{-\delta(T-\tau)}\,\Phi(-d_1)\right)e^{-\delta\tau}$$

where $d_1 = \dfrac{(r-\delta+0.5\sigma^2)(T-\tau)}{\sigma\sqrt{T-\tau}}$ and $d_2 = d_1 - \sigma\sqrt{T-\tau}$.

We can then see that the value of a forward start option is $e^{-\delta\tau}$ times the value of similar option starting today.

Forward start options can be packed together in what is known as a *cliquet option*, which consists in a series of call or put options with rules determining how the strike price is determined.

For example, a cliquet might consist of 20 at-the-money three-month options. The total life would then be five years. When one option expires a new similar at-the-money is coming into existence. As you can see, this would be a portfolio of 20 forward starting options that we just saw how to value.

Chooser Options

A *chooser option* allows the buyer to choose at an intermediate time before expiration whether to get a European call or put option. If the chooser option starts at time 0 and matures at T, then at time τ $(0 < \tau < T)$ the buyer chooses whether it is a put or call with strike K and expiring at T.

Therefore, the value of the chooser option at τ is $\max(C_\tau, P_\tau)$ since the buyer will choose to get whichever option is more valuable. From put-call parity we get that

$$P_\tau = C_\tau + Ke^{-r(T-\tau)} - S_\tau e^{-\delta(T-\tau)}$$

which implies that:

$$\max(C_\tau, P_\tau) = C_\tau + e^{-\delta(T-\tau)} \max\left(Ke^{-(r-\delta)(T-\tau)} - S_\tau, 0\right)$$

This is the payoff of a call with strike K and expiring at T plus $e^{-\delta(T-\tau)}$ puts with strike $\widetilde{K} = Ke^{-(r-\delta)(T-\tau)}$ and expiring at time τ.

Compound Options

We can also write options on other options, which are also known as *compound options*. We can then have four possible combinations:

- Call on call
- Put on call
- Call on put
- Put on put

These options can be valued analytically (we will not cover this in class, though). Intuitively, the price of such options is quite low compared with the underlying option.

Path-Dependent Options

The payoffs of some options might depend not only on the final value of the stock but on the full history of the stock price until maturity. These type of options are commonly called *path-dependent* options.

Lookback Options

The payoff of a *lookback option* depends on the maximum (S_{max}) or minimum (S_{min}) value of the stock until maturity. There are two type of lookback options, namely floating and fixed lookbacks.

A *floating lookback* call pays $S_T - S_{min}$ at time T. The idea is that the strike of the call is now replaced by the minimum value that the stock can have until maturity. Since by definition $S_{min} \leq S_T$, we always have that the payoff of the floating lookback call is non-negative. Similarly, the payoff of a floating lookback put is defined as $S_{max} - S_T$ and by construction is also non-negative.

A *fixed lookback* call is like a regular call whose final payoff depends on the maximum value of the stock during the lifetime of the option, that is, it pays $\max(S_{max} - K, 0)$ at maturity. A fixed lookback put pays $\max(K - S_{min}, 0)$ at maturity.

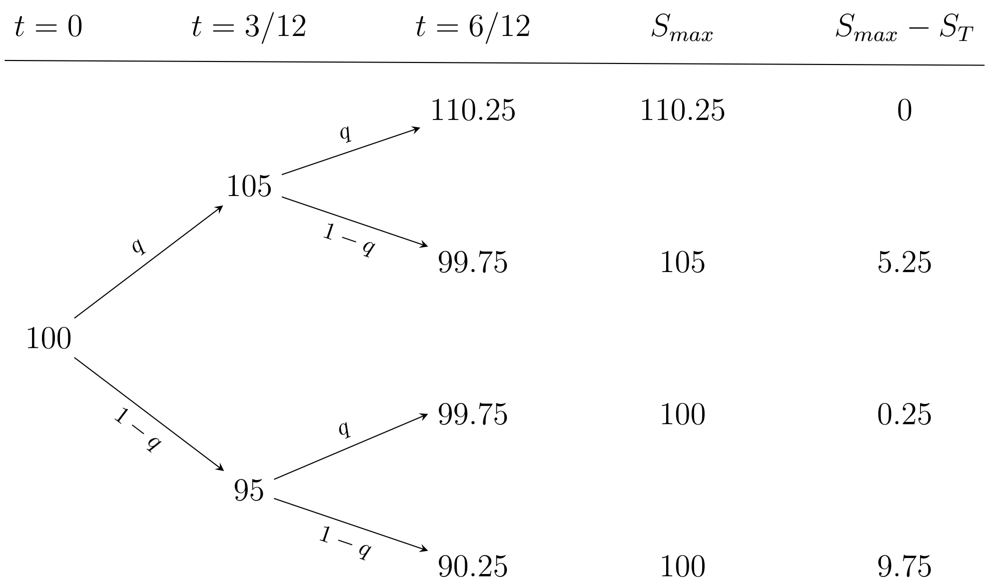
The table below summarizes the payoffs of the different looback options.

Type	Call	Put
Floating lookback	$S_T - S_{min}$	$S_{max} - S_T$
Fixed lookback	$\max(S_{max} - K, 0)$	$\max(K - S_{min}, 0)$

Even though is possible to derive analytic formulas for all types, we can easily price any lookback option by using a binomial tree as the next example shows.

Example 5. Consider a non-dividend paying stock that trades for \$100. Every 3-months, the stock price can increase or decrease by 5%. The risk-free rate is 6% per year with continuous compounding. We will compute the price of a floating lookback put that pays $S_{max} - S_T$ in 6 months.

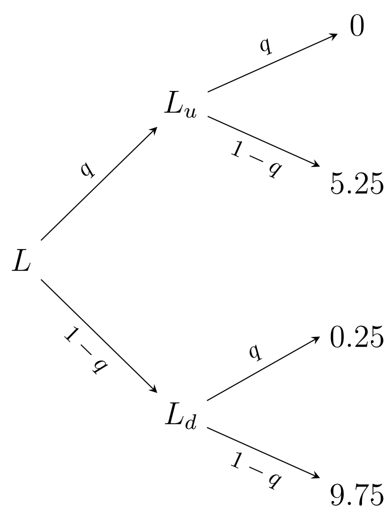
For lookback options is better to draw the full tree for the stock in order to see all four possible histories for the stock price. We can then compute the maximum stock price of each history and then the final payoff of the option, as the tree below shows.



The risk-neutral probability of an up-move is

$$q = \frac{e^{-0.06\times3/12} - 0.95}{1.05 - 0.95} = 0.6511$$

The tree for the lookbak put is presented below.



We can then compute:

$$L_u = (0q + 5.25(1-q))e^{-0.06\times3/12} = 1.80$$
$$L_d = (0.25q + 9.75(1-q))e^{-0.06\times3/12} = 3.51$$
$$L = (1.80q + 3.51(1-q))e^{-0.06\times3/12} = 2.36$$

Alternatively, we could obtain the price of the lookback put directly from the final payoffs:

$$L = (0q^2 + 5.25q(1-q) + 0.25(1-q)q + 9.75(1-q)^2)e^{-0.06\times 6/12} = 2.36$$

Asian Options

The payoff of an *Asian option* is related to the average stock price $\bar{S}$ from time 0 until T:

$$\bar{S}_T = \frac{1}{T}\int_0^T S_t dt$$

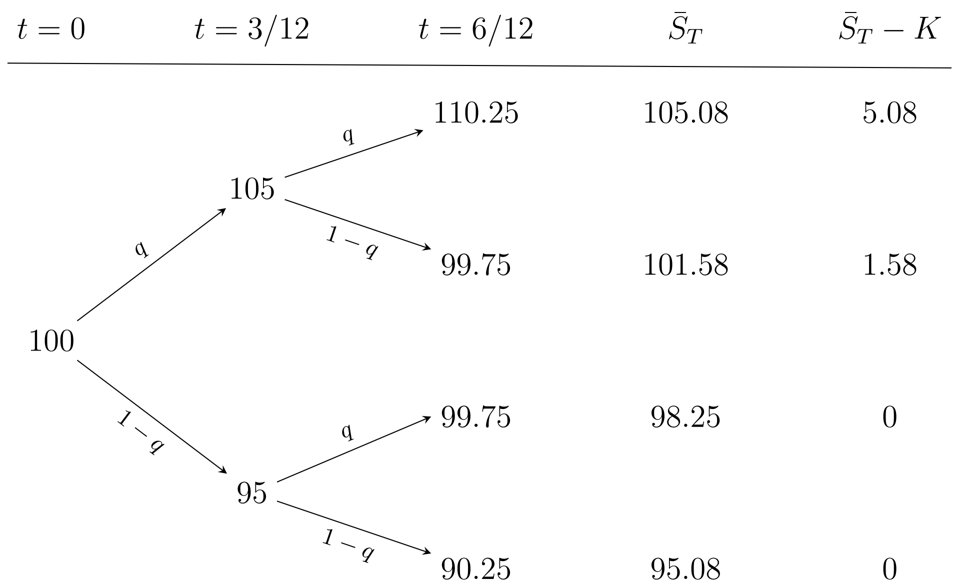
As with lookback options, there are two types of Asian options:

- An *average price* call pays max$(\bar{S}_T - K, 0)$ whereas an *average price* put pays max$(K - \bar{S}_T, 0)$ at maturity.
- An *average strike* call pays max$(S_T - \bar{S}_T, 0)$ whereas an *average strike* put pays max$(\bar{S}_T - S_T, 0)$

There is no exact analytic valuation for Asian options, but they can be approximately valued by assuming that the average stock price is lognormally distributed. Alternatively, it is straightforward to price them using a binomial tree.

Example 6. Consider a non-dividend paying stock that trades for $100. Every 3-months, the stock price can increase or decrease by 5%. The risk-free rate is 6% per year with continuous compounding. We will compute the price of an average price call that pays $\bar{S}_T - K$ in 6 months, where $K = 100$.

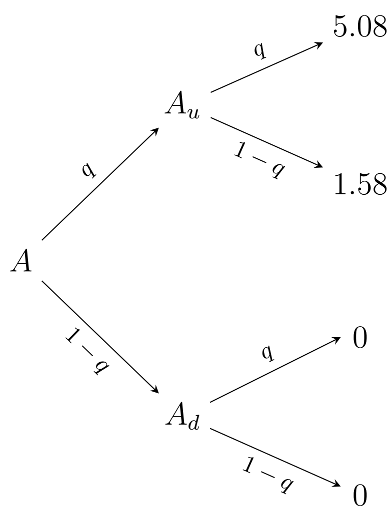
As we did for lookback options, it is better to draw the full tree for the stock in order to see all four possible histories for the stock price. We can then compute the average stock price of each history and then the final payoff of the option, as the tree below shows.



The risk-neutral probability of an up-move is

$$q = \frac{e^{-0.06\times 3/12} - 0.95}{1.05 - 0.95} = 0.6511$$

The tree for the Asian call is presented below.



We can then compute:

$$A_u = (5.08q + 1.58(1-q))e^{-0.06\times 3/12} = 3.80$$
$$A_d = (0q + 0(1-q))e^{-0.06\times 3/12} = 0$$
$$A = (3.80q + 0(1-q))e^{-0.06\times 3/12} = 2.44$$

Alternatively, we could obtain the price of the lookback put directly from the final payoffs:

$$L = (5.08q^2 + 1.58q(1-q) + 0(1-q)q + 0(1-q)^2)e^{-0.06\times 6/12} = 2.44$$

Barrier Options

Barrier options are either call or put options that get activated or deactivated depending on whether the stock hits a barrier from above or below.

In options come into existence only if stock price hits the barrier before option maturity. *Out* options die if stock price hits the barrier before option maturity.

Up options require that the stock hits the barrier from below, whereas *down* options require that the stock hit the barrier from above.

Therefore, there are eight possible combinations.

Example 7. Consider an *in-and-down* call with barrier \$90, strike price \$110, and expiring in 6 months. The stock price is currently \$100.

- Case 1: During the next six months, the stock goes down to \$85 and but then finishes at \$120. The in-and-out call pays $120 - 110 = \$10$.
- Case 2: During the next six months, the stock never goes below \$90 and finishes at \$140. The in-and-out call pays nothing since even though it is in-the-money it never gets activated.
- Case 3: During the next six months, the stock goes down to \$85 and and finishes at \$105. Even though the in-and-out call gets activated, it is out-of-the-money at maturity and hence pays nothing.

Other Exotic Options

Exchange Options

An *exchange option* is an option to exchange one asset for another. For example, an option to exchange one unit of U for one unit of V.The payoff is then $\max(V_T - U_T, 0)$.

In order to price such an option we need to know how the two assets are correlated. It is hard to value such an option using a binomial tree since we need to take into account the correlation of the two assets. It is possible, however, to find an analytical formula using the Black & Scholes framework.

Basket Options

A *basket option* is an option to buy or sell a portfolio of assets. This can be valued by calculating the first two moments of the value of the basket at option maturity and then assuming it is lognormal.

Non-Standard American Options

In practice, many American type options have restrictions about when they can be exercised. Some American options are exercisable only on specific dates (Bermudans), whereas other options can be exercised early only after a certain *lock out* period has elapsed. For some

Some American options have a strike price that changes over the life of the contract, such as warrants or convertibles.

Practice Problems

Solutions to all problems can be found at lorenzonaranjo.com/fin451.

Problems 1 to 6 are based on an asset with the following characteristics:

Variable	Value
S	100
r	10%
δ	4%
σ	30%

Problem 1. Compute the price of a long range forward with maturity 9 months, and strikes $K_1 = \$90$ and $K_2 = \$120$.

Problem 2. Compute the price of a gap put option with maturity 9 months, that pays $90 - S_T$ whenever $S_T < 110$.

Problem 3. Compute the price of a forward start put option that starts in 9 months, with final maturity in 15 months, and that is struck at-the-money when it starts.

Problem 4. Compute the price of a chooser option with maturity 15 months, strike $K = \$110$, and where the buyer can choose whether is a call or a put in 6 months.

Problem 5. Compute the price of a cash-or-nothing binary option with maturity 9 months that pays $50 whenever $S_T > 100$.

Problem 6. Compute the price of an asset-or-nothing binary option with maturity 9 months that pays one unit of the stock whenever $S_T > 100$.

Problem 7. Consider a non-dividend paying stock that trades for $100. Every 3-months, the stock price can increase or decrease by 10%. The risk-free rate is 10% per year with continuous compounding. Compute the price of the following path-dependent options expiring in 6 months:

a. A floating lookback put that pays $S_{max} - S_T$ at maturity.
b. A fixed lookback call that pays $\max(S_{max} - 100, 0)$ at maturity.
c. An average price Asian call option that pays $\max(\bar{S} - 100, 0)$ at maturity.
d. A floating lookback call that pays $S_T - S_{min}$ at maturity.