The Greeks

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Summary

- In the Black-Scholes model where

$$dS = (r - q)Sdt + \sigma S dW$$

Greek Letter	Call	Put
Value (V)	$Se^{-qT}\,\Phi(d_1) - Ke^{-rT}\,\Phi(d_2)$	$Ke^{-rT}\,\Phi(-d_2) - Se^{-qT}\,\Phi(-d_1)$
Delta (Δ)	$e^{-qT}\,\Phi(d_1)$	$-e^{-qT}\,\Phi(-d_1)$
Gamma (Γ)	$\frac{e^{-qT}\,\Phi'(d_1)}{S\sigma\sqrt{T}} = \frac{Ke^{-rT}\,\Phi'(d_2)}{S^2\sigma\sqrt{T}}$	
Theta (Θ)	$rV - (r-q)S\Delta - \frac{1}{2}\sigma^2 S^2\Gamma$	
Vega ($\mathcal{V}$)	$Se^{-qT}\,\Phi'(d_1)\sqrt{T} = Ke^{-rT}\,\Phi'(d_2)\sqrt{T}$	
Rho (ρ)	$KTe^{-rT}\,\Phi(d_2)$	$-KTe^{-rT}\,\Phi(-d_2)$

Outline

1. Delta Hedging
2. Delta, Theta and Gamma
3. Vega and Rho

Delta Hedging

- In many cases (but not always), the seller of an option might want to hedge a position dynamically, i.e. by rebalancing a portfolio consisting in the stock and a risk-free bond.
- Even though static hedging might be desirable (like buying another option), it might not be feasible or economically viable.
- In such cases, it is useful to think about how to replicate the option dynamically.
- We call this process delta hedging, and the resulting portfolio is said to be **delta neutral**.

Delta of a Portfolio

- The delta of an option (or portfolio) measures its sensitivity to the underlying asset price.
- Formally, if we denote by V the value of a portfolio (possibly containing the stock, risk-free bonds and options), the delta of the portfolio is defined as:

$$\Delta = \frac{\partial V}{\partial S}$$

- The delta of a European call or put option can be computed from the Black-Scholes formula as shown before. The process of **delta hedging** involves buying or selling stocks as determined by the delta.

Example 1

Consider a non-dividend paying stock that currently trades at \$50. A trader just sold 50 call option contracts (5,000 options) written on the stock. The current option price is \$4.13 and the option's delta is 0.591. The money-market risk-free rate is 5% per year with simple compounding.

First hedge: The delta of the position is $-5{,}000(0.591) = -2{,}955$. To delta hedge the position, the trader buys 2,955 shares for a cost of \$147,750. By writing the calls, the trader receives $5{,}000(4.13) = \$20{,}650$, which amounts to a net expense of \$127,100 that the trader borrows. Since the trader is delta neutral, the interest rate on the loan is the risk-free rate.

Example 1 (Continued)

Price Change: During the next week, the stock price increases to \$50.53, the option price increases to \$4.35, and the delta changes to 0.612. The delta of the option position changes to $-5{,}000(0.612) = -3{,}060$.

Profit & Loss: The long stock position is now worth $2{,}955(50.53) = \$149{,}316.15$, whereas the short call position is worth $-5{,}000(4.35) = -\$21{,}750$. The loan now accrues to $-127{,}100\left(1 + \frac{0.05}{52}\right) = -\$127{,}222.21$. The weekly P&L is then $149{,}316.15 - 21{,}750 - 127{,}222{,}21 = \$343.94$, which is a small percentage of the total exposure. We will keep this extra money in a separate account.

Hedge rebalancing: The trader buys an additional $3{,}060 - 2{,}955 = 105$ shares to maintain delta neutrality. The total cost of the new long position in shares is $3{,}060(50.53) = \$154{,}621.80$. The trader needs to borrow in total $154{,}621.80 - 21{,}750 = \$132{,}871.80$, or an additional $132{,}871.80 - 127{,}222{,}21 = \$5{,}649.59$.

Example 2

Consider a non-dividend paying stock that currently trades at \$50. The money-market risk-free rate is 5% per year with continuous compounding, and the volatility of log-returns is 25% per year. Compute the delta of a European call option with maturity 6 months and strike of \$50.

$$d_1 = \frac{\ln(50/50) + (0.05 + \frac{1}{2}(0.25)^2)(0.5)}{0.25\sqrt{0.5}} = 0.2298$$

The delta is then equal to $\Phi(d_1) = 0.591$.

Example 3

Consider:

1. A long position in 100,000 call options (1,000 contracts) with strike price \$100 and expiration in 9 months. The delta of each option is 0.597.
2. A short position in 200,000 call options (2,000 contracts) with strike \$110 and expiration in 6 months. The delta of each option is 0.368.
3. A short position in 50,000 put options (500 contracts) with strike \$90 and expiration in 3 months. The delta of each option is -0.162.

The delta of the portfolio is:

$$100{,}000(0.597) - 200{,}000(0.368) - 50{,}000(-0.162) = -5{,}800$$

The portfolio can then be made delta neutral by **buying** 5,800 shares.

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Theta of a Portfolio

- The theta (Θ) of a portfolio (V) captures the rate of change in value of that portfolio with respect to the passage of time, i.e.

$$\Theta = \frac{\partial V}{\partial t}$$

- The theta is also referred as the **time-decay** of the portfolio and is usually monitored by traders since it is a good proxy for gamma in a delta neutral portfolio.
- For European call and put options we have that:

$$\Theta_C = -e^{-qT} \frac{S\,\Phi^{'}(d_1)\sigma}{2\sqrt{T}} - rKe^{-rT}\,\Phi(d_2) + qSe^{-qT}\,\Phi(d_1)$$
$$\Theta_P = -e^{-qT} \frac{S\,\Phi^{'}(d_1)\sigma}{2\sqrt{T}} + rKe^{-rT}\,\Phi(-d_2) - qSe^{-qT}\,\Phi(-d_1)$$

Gamma of a Portfolio

- The gamma (Γ) of a portfolio measures the rate of change of the portfolio's delta with respect to the price of the underlying asset, i.e.

$$\Gamma = \frac{\partial \Delta}{\partial S} = \frac{\partial^2 V}{\partial S^2}$$

- When gamma is small, delta changes slowly and the portfolio is kept delta neutral without many changes.
- On the other hand, when gamma is high, it is important to monitor the portfolio frequently and adjust delta neutrality as needed.
- The gamma for European call and put options is:

$$\Gamma_C = \Gamma_P = \frac{e^{-qT}\,\Phi'(d_1)}{S\sigma\sqrt{T}} = \frac{Ke^{-rT}\,\Phi'(d_2)}{S^2\sigma\sqrt{T}}$$

Gamma and Delta Neutrality

- Remember that according to Ito's Lemma:

$$dV = \frac{\partial V}{\partial S} dS + \frac{1}{2} \frac{\partial^2 V}{\partial S^2} (dS)^2 + \frac{\partial V}{\partial t} dt$$

or

$$dV = \Delta dS + \frac{1}{2} \Gamma (dS)^2 + \Theta dt$$

- Therefore, in a delta-neutral portfolio we have that:

$$dV = \frac{1}{2} \Gamma (dS)^2 + \Theta dt$$

which implies that:

$$\Delta V \approx \frac{1}{2} \Gamma (\Delta S)^2 + \Theta \Delta t$$

Example 4

Suppose that the gamma of a delta-neutral portfolio is 10,000. A jump of $+\$2$ or $-\$2$ in the underlying asset will approximately increase the value of the portfolio by $\frac{1}{2}10{,}000(2)^2 = \$20{,}000$.

Example 5

A trader's portfolio is delta-neutral and has a gamma of 5,000. The delta and gamma of a traded call option is 0.52 and 1.60, respectively. The trader wants to make the portfolio both delta and gamma-neutral.

The portfolio can be made gamma-neutral by selling $\frac{5{,}000}{1.60} = 3{,}125$ call options. The portfolio can now be made delta-neutral by buying $0.52(3{,}125) = 1{,}625$ shares of the underlying asset. Note that the shares have zero gamma, so they do not change the gamma of the portfolio but only affect its delta.

Example 6

Let's compute the gamma of the option in Example 2.

$$d_1 = \frac{\ln(50/50) + (0.05 + \frac{1}{2}(0.25)^2)(0.5)}{0.25\sqrt{0.5}} = 0.2298$$

$$\Phi^{'}(d_1) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(0.2298)^2} = 0.3885$$

$$\Gamma = \frac{\Phi^{'}(d_1)}{S_0 \sigma \sqrt{T}} = \frac{0.3885}{50(0.25)\sqrt{0.5}} = 0.0440$$

Gamma and Theta

- Remember the fundamental Black-Scholes differential equation:

$$(r-q)S\frac{\partial V}{\partial S}+\frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2}+\frac{\partial V}{\partial t}=rV$$

which can be re-written using the Greeks as:

$$(r-q)S\Delta+\frac{1}{2}\sigma^2 S^2\Gamma+\Theta=rV$$

- Hence, for a delta-neutral portfolio we have that:

$$\frac{1}{2}\sigma^2 S^2\Gamma+\Theta=rV$$

- Therefore, when theta is large and positive, the gamma of a portfolio tends to be large and negative, and vice-versa.

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Vega

- The volatility that gives the right price of the option under the Black-Scholes is called the implied volatility.
- The sensitivity of the option to its implied volatility is called vega:

$$\mathcal{V} = \frac{\partial V}{\partial \sigma}$$

- Usually vega risk is more relevant for longer maturity options, whereas gamma risk is more prominent for shorter maturity options.

$$\mathcal{V}_C = \mathcal{V}_P = Se^{-qT}\,\Phi'(d_1)\sqrt{T} = Ke^{-rT}\,\Phi'(d_2)\sqrt{T}$$

Example 7 (From Hull)

Consider a portfolio that is delta neutral, with a gamma of -5,000 and a vega of -8,000. The options shown in the table below can be traded.

	Delta	Gamma	Vega
Portfolio	0	-5,000	-8,000
Option 1	0.6	0.5	2.0
Option 2	0.5	0.8	1.2

The portfolio can be made vega neutral by including a long position in 4,000 of Option 1. This would increase delta to 2,400 and require that 2,400 units of the asset be sold to maintain delta neutrality. The gamma of the portfolio would change from -5,000 to -3,000.

Example 7 (From Hull, Continued)

To make the portfolio both gamma and vega neutral, both Option 1 and Option 2 can be used. We must then solve:

$$-5{,}000 + 0.5N_1 + 0.8N_2 = 0$$
$$-8{,}000 + 2.0N_1 + 1.2N_2 = 0$$

These yields $N_1 = 400$ and $N_2 = 6{,}000$. The new delta is $400(0.6) + 6{,}000(0.5) = 3{,}240$. Hence, we sell 3,240 units of the underlying asset to maintain delta-neutrality.

Rho of a Portfolio

- The rho (ρ) of a portfolio measures the rate of change of the portfolio's value with respect to the risk-free rate, i.e.

$$\rho = \frac{\partial V}{\partial r}$$

- The rho for a European call and put options is:

$$\rho_C = KTe^{-rT}\,\Phi(d_2) > 0$$
$$\rho_P = -KTe^{-rT}\,\Phi(-d_2) < 0$$

- The rho of a call is positive since it is a levered position in the risk-free asset whereas the rho of the put is negative since it borrows the stock to invest in the risk-free asset.