The Black-Scholes Model

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Pricing Formulas

Black-Scholes Model for a Non-Dividend Paying Stock

Consider a non-dividend paying stock S that follows a GBM under the risk-neutral measure:

$$dS = rSdt + \sigma SdW$$

The price of European call and put options with strike price K and time-to-maturity T are given by:

$$C = S\,\Phi(d_1) - Ke^{-rT}\,\Phi(d_2)$$
$$P = Ke^{-rT}\,\Phi(-d_2) - S\,\Phi(-d_1)$$

where

$$d_1 = \frac{\ln(S/K) + (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$$
$$d_2 = d_1 - \sigma\sqrt{T}$$

The Replicating Portfolio Approach

- Consider a derivative V written on a non-dividend paying stock S with maturity T that pays $F(S)$ at maturity.
- The binomial model implies that the derivative can be replicated by buying (or selling) α_t units of the stock and β_t units of a bond with face value K and maturity T, respectively.
- If we call V the value of such replicating portfolio, we have that at time $t < T$:

$$V_t = \alpha_t S_t + \beta_t B_t.$$

- In order to replicate the derivative, we want to make sure that the value of the portfolio at time $t = T$ is equal to the payoff of the derivative, that is:

$$V_T = H_T$$

- For example, for a European call option $H_T = \max(S_T - K, 0)$.

The Replicating Portfolio is Self-Financing

- At time $t + \Delta t$, the value of the replicating portfolio is:

$$V_{t+\Delta t} = \alpha_t S_{t+\Delta t} + \beta_t B_{t+\Delta t},$$

which implies that:

$$\Delta V_t = \alpha_t \Delta S_t + \beta_t \Delta B_t.$$

- The new composition of the portfolio at time $t + \Delta t$ is chosen such that:

$$V_{t+\Delta t} = \alpha_t S_{t+\Delta t} + \beta_t B_{t+\Delta t} = \alpha_{t+\Delta t} S_{t+\Delta t} + \beta_{t+\Delta t} B_{t+\Delta t}$$

which shows that the portfolio is self-financing, i.e., no new funds are added or withdrawn from the portfolio.

Replication in Continuous-Time

- As $\Delta t \to 0$, we have that:

$$\begin{aligned} dV &= \alpha dS + \beta dB \\ &= \alpha dS + \beta(rBdt) \\ &= \alpha dS + (\beta B)rdt \end{aligned}$$

- And since $V = \alpha S + \beta B \Rightarrow \beta B = V - \alpha S$, we can conclude that:

$$dV = r(V - \alpha S)dt + \alpha dS$$

Applying Ito's Lemma

- We will assume for the moment that V is a smooth function of S and t, that is, $V = V(S, t)$.
- Then, Ito's Lemma implies that:

$$
\begin{aligned}
dV &= \frac{\partial V}{\partial S} dS + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}(dS)^2 + \frac{\partial V}{\partial t} dt \\
&= \left(\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t}\right) dt + \frac{\partial V}{\partial S} dS
\end{aligned}
$$

- Therefore:

$$
\left(\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t}\right) dt + \frac{\partial V}{\partial S} dS = r(V - \alpha S)dt + \alpha dS
$$

The Delta of the Derivative

- First, the previous equation shows that replication works if and only if:

$$\alpha = \frac{\partial V}{\partial S}$$

- This is a fundamental relationship in derivatives pricing.
- It states that the number of shares needed to replicate the derivative is equal its sensitivity to the underlying asset.
- We call this quantity the delta (Δ) of the derivative.
- Also, note that by choosing α equal to the delta of the derivative, it really does not matter what drift we have for the stock.
- We will use this fact in a moment to define the risk-neutral probabilities in continuous-time.

The Fundamental Partial Differential Equation (PDE)

- Second, it must be the case that:

$$\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t} = r\left(V - S\frac{\partial V}{\partial S}\right)$$

- Therefore:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0$$

  subject to $V_T = H_T$.

- This is the celebrated Black-Scholes partial differential equation (PDE) which allowed the authors to compute their influential formula in 1973!
- Solving PDEs, in general, is very hard so we will resort to a different approach to price European call and put options.

The Risk-Neutral Pricing Approach

- The replicating approach is insensitive to the drift of the stock.
- As a matter of fact, the drift might even change based on whose thinking about the asset.
- Since the previous reasoning is silent about the drift and the type of investor pricing the asset, we can assume in our reasoning that all investors are **risk-neutral**.
- Even if this is not true in real markets, such assumption would not affect **the logic** of the replicating-portfolio argument.

A Risk-Neutral World

- In a world populated by risk-neutral investors, the price today of any non-dividend paying asset is equal to the expected payoff at maturity discounted at the risk-free rate, that is:

$$X = e^{-rT}\,\mathrm{E}(X_T)$$

- Therefore, the drift of a non-dividend paying stock is the risk-free rate:

$$dS = rSdt + \sigma SdW$$

- The same is true for all derivatives written on the stock:

$$dV = \underbrace{\left(rS\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t}\right)}_{=rV} dt + \left(\sigma S \frac{\partial V}{\partial S}\right) dW$$

- We recover the same equation as before!

Pricing a European Call Option

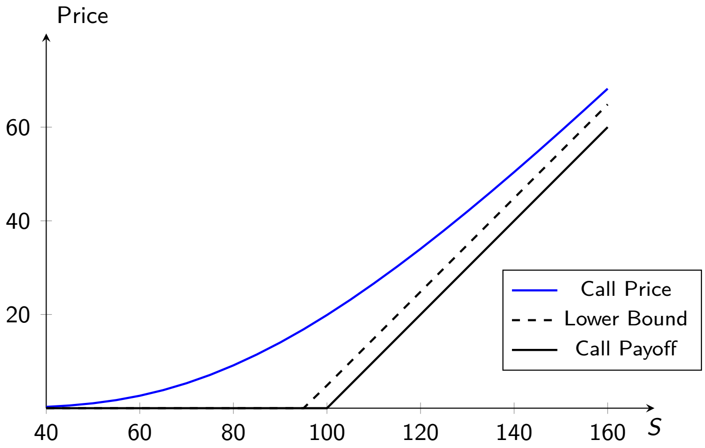
- Consider a European call option written on a non-dividend paying stock with maturity T and strike price K.
- The price of the call should then be:

$$
\begin{aligned}
C &= e^{-rT}\,\mathrm{E}\left((S_T - K)\mathbb{1}_{\{S_T > K\}}\right) \\
&= e^{-rT}\,\mathrm{E}\left(S_T\,\mathbb{1}_{\{S_T > K\}}\right) - e^{-rT}\,\mathrm{E}\left(K\,\mathbb{1}_{\{S_T > K\}}\right) \\
&= S\,\Phi(d_1) - Ke^{-rT}\,\Phi(d_2)
\end{aligned}
$$

where

$$
\begin{aligned}
d_1 &= \frac{\ln(S/K) + (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} \\
d_2 &= d_1 - \sigma\sqrt{T}
\end{aligned}
$$

Call Premium vs. Spot Price



The plot displays the Black-Scholes call premium $C(S)$ where $r = 0.05$, $\sigma = 0.45$, $T = 1$ and $K = 100$. It also shows the call option payoff given by $\max(S - 100, 0)$ and the lower bound for a European call given by $\max(S - 100e^{-0.05(1)}, 0)$.

Reconciling Both Pricing Approaches

- It is tedious but straightforward to prove that:

$$\alpha = \frac{\partial C}{\partial S} = \Phi(d_1) \tag{1}$$

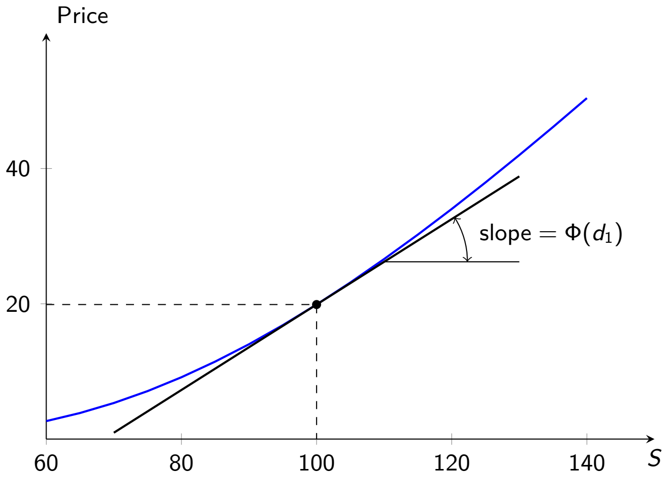
- Also, we have that for a European call option:

$$C = \alpha S + \beta B = S\,\Phi(d_1) - Ke^{-rT}\,\Phi(d_2)$$

which because of (1) implies that:

$$\beta = -\,\Phi(d_2)$$

Call Delta



The figure plots the Black-Scholes call premium $C(S)$ where $r = 0.05$, $\sigma = 0.45$, $T = 1$ and $K = 100$, and shows the tangent line at $S = 100$ whose slope coefficient is the delta of the call given by $\Phi(d_1)$.

Hedging the Call

- Our analysis so far implies that to replicate a European call option, we need to go *long* $\Phi(d_1)$ shares of stock and *short* $\Phi(d_2)$ risk-free bonds with face value K and maturity T.
- The call is therefore a levered position in the underlying asset whose delta is given by $\Phi(d_1)$.
  - Since $0 < \Phi(d_1) < 1$, the delta of the call for a non-dividend paying asset is bounded between 0 and 1.
- As we saw in the previous slide, for a given spot price, the delta of the call represents the slope coefficient of the tangency line at that point.

Pricing a European Put Option

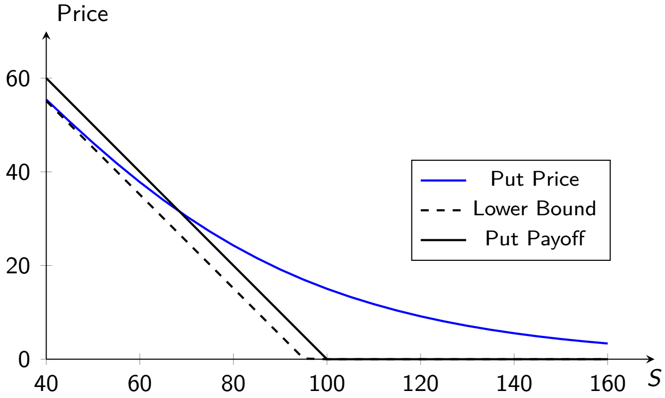
- Consider now a European put option with the same characteristics as the previous call.
- According to put-call parity, it must be the case that:

$$C - P = S_0 - Ke^{-rT}$$

- Hence,

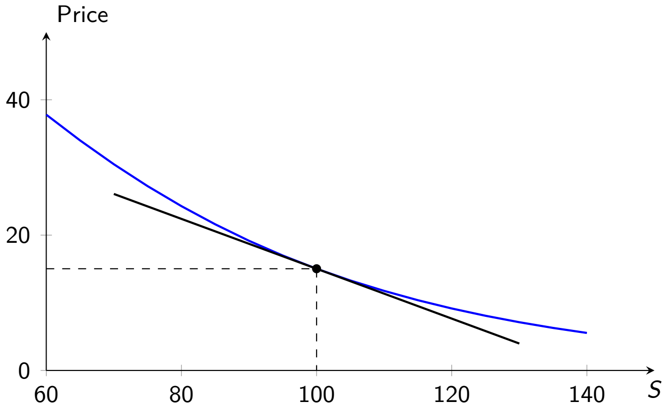
$$\begin{aligned} P &= C - (S - Ke^{-rT}) \\ &= S\,\Phi(d_1) - Ke^{-rT}\,\Phi(d_2) - (S - Ke^{-rT}) \\ &= Ke^{-rT}(1 - \Phi(d_2)) - S(1 - \Phi(d_1)) \\ &= Ke^{-rT}\,\Phi(-d_2) - S\,\Phi(-d_1) \end{aligned}$$

Put Premium vs. Spot Price



The plot displays the Black-Scholes put premium $P(S)$ where $r = 0.05$, $\sigma = 0.45$, $T = 1$ and $K = 100$. It also shows the put option payoff given by $\max(100 - S, 0)$ and the lower bound for a European put given by $\max(100e^{-0.05(1)} - S, 0)$.

Put Delta



The figure plots the Black-Scholes put premium $P(S)$ where $r = 0.05$, $\sigma = 0.45$, $T = 1$ and $K = 100$, and shows the tangent line at $S = 100$ whose slope coefficient is the delta of the put given by $-\Phi(-d_1) = \Phi(d_1) - 1$.

Hedging the Put

- We can use put-call parity to compute α for the put:

$$\alpha = \frac{\partial P}{\partial S} = \frac{\partial(C - S + Ke^{-rT})}{\partial S} = \Phi(d_1) - 1 = -\Phi(-d_1) < 0$$

- The fact that we also have $P = \alpha S + \beta B$ also implies that:

$$\beta = \Phi(-d_2) > 0$$

- Therefore, to replicate a European put option, we need to go *short* $\Phi(-d_1)$ shares of stock and *long* $\Phi(-d_2)$ risk-free bonds with face value K and maturity T.

Finishing In-The-Money

- Remember that we showed that:

$$\Pr(S_T > K) = \mathrm{E}\left(\mathbb{1}_{\{S_T > K\}}\right) = \Phi(d_2)$$

which also implies that:

$$\Pr(S_T < K) = 1 - \Pr(S_T > K) = 1 - \Phi(d_2) = \Phi(-d_2)$$

- Therefore, the risk-neutral probability that the call will expire in-the-money is equal to $\Phi(d_2)$ whereas the risk-neutral probability that the put finishes in-the-money is given by $\Phi(-d_2)$.

Example 1

Consider a non-dividend paying stock that currently trades for $100. The risk-free rate is 4% per year, continuously compounded and constant for all maturities. The instantaneous volatility of returns is 25% per year. Consider at-the-money call and put options written on the stock with maturity 9 months. Then,

$$d_1 = \frac{\ln(100/100) + (0.04 + 0.5(0.25)^2)(0.75)}{0.25\sqrt{0.75}} = 0.2468$$

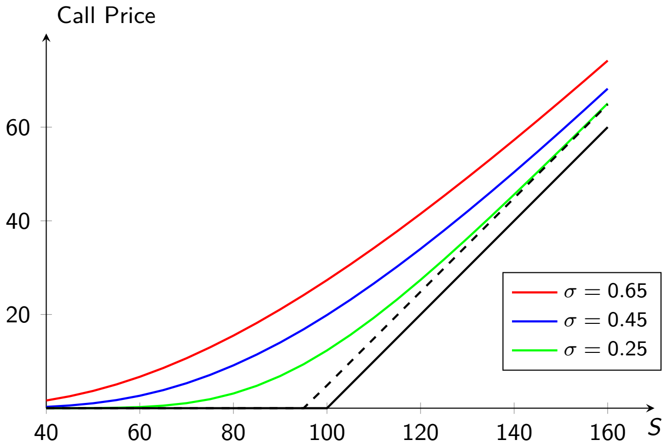
$$d_2 = 0.2468 - 0.25\sqrt{0.75} = 0.0303$$

Therefore, $\Phi(d_1) = 0.5975$ and $\Phi(d_2) = 0.5121$, which implies that:

$$C = 100(0.5975) - 100e^{-0.04(0.75)}(0.5121) = \$10.05$$

$$P = 100e^{-0.04(0.75)}(1 - 0.5121) - 100(1 - 0.5975) = \$7.10$$

The Impact of Volatility



The figure shows the Black-Scholes call premium for different levels of volatility where $r = 0.05$, $T = 1$ and $K = 100$. The dashed line represents the lower bound for the European call and the solid black line is the call payoff at maturity.

Option Premium vs. Volatility

- One of the most important determinants of option prices in the Black-Scholes model is volatility.
- We can show that for European call and put options:

$$\frac{\partial C}{\partial \sigma} = \frac{\partial P}{\partial \sigma} = S\,\Phi^{'}(d_1)\sqrt{T} > 0 \tag{2}$$

- Hence, both European call and put options increase in value as volatility increases.
- Moreover, this also implies that there is a one-on-one relationship between option value and volatility, i.e., we can use volatility to quote prices and vice-versa.
- The volatility that matches the observed price of an option is called the **implied volatility**.

Implied Volatility

Example 2

Consider a non-dividend paying stock that currently trades for \$100. The risk-free rate is 5% per year, continuously compounded and constant for all maturities. An ATM European call option written on the stock with maturity 12 months trades for \$16. We can check that $\sigma = 34.66\%$ prices the call correctly:

$$d_1 = \frac{\ln(100/100) + (0.05 + 0.5(0.3466)^2)(1)}{0.3466\sqrt{1}} = 0.3176$$
$$d_2 = 0.3358 - 0.3466\sqrt{1} = -0.0290$$

Therefore, $\Phi(d_1) = 0.6246$ and $\Phi(d_2) = 0.4884$, which implies that

$$C = 100(0.6246) - 100e^{-0.05(1)}(0.4884) = \$16.00$$

How Can We Compute the Implied Volatility?

- Unfortunately, it is not possible to solve analytically for the implied volatility.
- For a call option, for example, it involves solving numerically for σ:

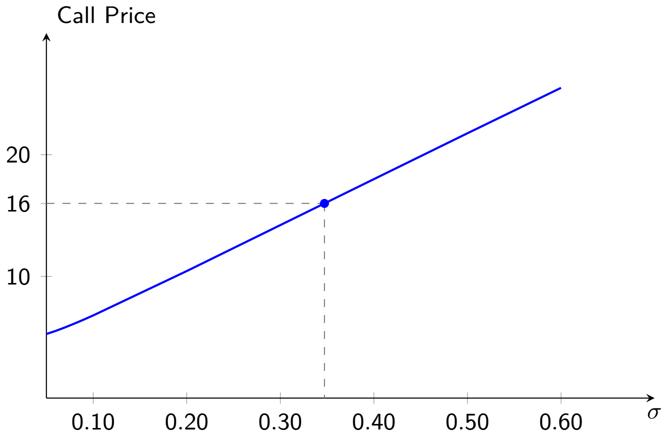
$$C_0 = C(\sigma_{imp})$$

- Alternatively, we could tabulate the price of a call option for different values of σ (using the same parameters as the previous example):

σ	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40
C	5.28	6.80	8.59	10.45	12.34	14.23	16.13	18.02

- We could see that $\sigma = 35\%$ gives a price of \$16.13 for the call, which is quite close to the true implied volatility of 34.66%.

Implied Volatility for a Call Option



The figure shows the Black-Scholes call premium $C(S, T; r, \sigma, K)$ as a function of σ where $S = 100$, $r = 0.05$, $T = 1$ and $K = 100$. We can see that for $C = \$16$ the corresponding volatility is approximatively 35%.