Modeling Stock Prices in Continuous-Time

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Stochastic Process

- A stochastic process describes the evolution of a random variable over time.
- In finance we use stochastic processes to model the evolution of stock prices, interest rates, volatility, foreign exchange rates, commodity prices, etc.
- We distinguish between:
  - **Discrete-time processes**: The values of the process $\{S_n\}$ are allowed to change only at discrete time intervals, i.e. $n \in \{0, 1, 2, \ldots, N\}$ or $n \in \mathbb{N}$.
  - **Continuous-time processes**: The stochastic process $\{S_t\}$ is defined for all $t \in [0, T]$.

Random Walk

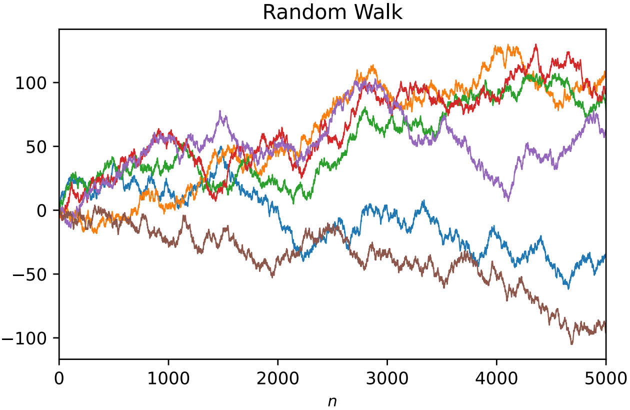
- A random walk $\{X_n\}$ is a stochastic process defined as:

$$X_0 = x_0$$
$$X_{n+1} = X_n + e_{n+1}$$

  where $\{e_n\}$ are independent and identically distributed (i.i.d.) random variables such that $\mathrm{E}(e_n) = 0$ for all $n \geq 1$.
- Note that e_n need not be normally distributed.
- For example, e_n could be such:

$$\Pr(e_n = 1) = \Pr(e_n = -1) = 0.5$$

Random Walk Simulation



The figure plots simulated paths for the random walk $X_{n+1} = X_n + e_{n+1}$ where $X_0 = 0$, $\{e_n\}$ is an i.i.d sequence taking the values 1 and -1 with equal probability, and $n \leq 5000$.

Martingales

- A discrete-time martingale $\{Z_n\}_{n\geq 0}$ is a stochastic process such that:

$$\mathrm{E}\left(Z_{n+1} \mid Z_1, Z_2, \ldots, Z_n\right) = Z_n$$

- Note that a martingale need not be a random walk.
  - For example, consider the process $\{Z_n\}$:

$$Z_{n+1} = Z_n \varepsilon_{n+1}$$

    where $\{\varepsilon_n\}$ is an i.i.d. sequence such that $\mathrm{E}\left(\varepsilon_n\right) = 1$ for all $n \geq 1$.
  - It is a martingale since:

$$\mathrm{E}\left(Z_{n+1} \mid Z_1, Z_2, \ldots, Z_n\right) = \mathrm{E}\left(Z_n \varepsilon_{n+1} \mid Z_n\right) = Z_n \, \mathrm{E}\left(\varepsilon_{n+1} \mid Z_n\right) = Z_n.$$

- Random walks are martingales, though.

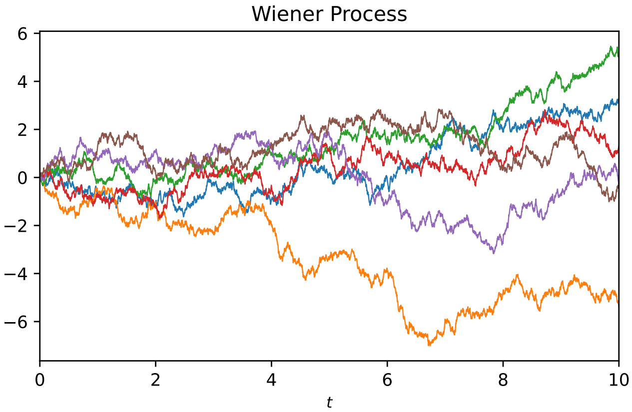
Wiener Process

- A very useful random walk can be defined as follows:

$$W_{t+\Delta t} = W_t + \sqrt{\Delta t} e_{t+\Delta t}$$

  where $W_0 = 0$ and $\{e_t\}$ are i.i.d. such that $e_t \sim N(0, 1)$.
- Note that here time increases each step by Δt.
- Letting $\Delta t \to 0$, the resulting process $\{W_t\}$ for $t \in [0, T]$ is called a Wiener process or Brownian motion.
- The Wiener process has the following properties:
  - The sample paths are continuous.
  - For $s < t$, the increment $W_t - W_s \sim N(0, t - s)$, i.e. is normally distributed with mean 0 and variance $t - s$.
  - Increments are independent of each other.
  - In particular, note that $W_t \sim N(0, t)$ for $0 < t \leq T$.

Wiener Process Simulation



The figure plots simulated paths for $\{W_t\}$ where $t \in [0, 10]$.

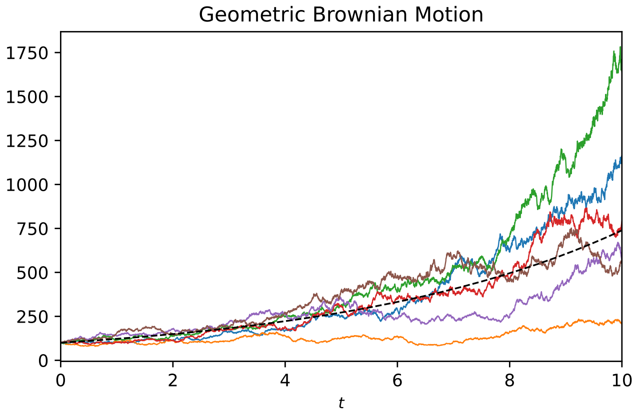
Geometric Brownian Motion

- Now we turn our attention to modeling stock prices $\{S_t\}$.
  - We need to be careful, though, as stock prices cannot be negative.
  - We also would like to allow the model to display a certain drift μ and volatility σ.
- To achieve this, we model the percentage change of a stock price between t and $t + \Delta t$ as:

$$\frac{\Delta S_t}{S_t} = \mu \Delta t + \sigma \Delta W_t$$

- Note that the percentage change in price over an interval Δt is normally distributed with mean $\mu \Delta t$ and variance $\sigma^2 \Delta t$.
- This process is called a geometric Brownian motion (GBM).

Geometric Brownian Motion Simulation



The figure plots simulated paths for a geometric Brownian motion $\{S_t\}$ where $t \in [0, 10]$, $S_0 = 100$, $\mu = 0.20$, and $\sigma = 0.20$. The dashed line denotes $\mathrm{E}(S_t) = S_0 e^{\mu t}$.

Preliminary Results on Wiener Processes

- The Wiener process increment can be approximated as:

$$\Delta W_t = W_{t+\Delta t} - W_t = \sqrt{\Delta t} e_{t+\Delta t}$$

- If we define $\xi = (\Delta W_t)^2$, we have that:

$$\begin{aligned} \mathrm{E}(\xi) &= \Delta t \\ \mathrm{V}(\xi) &= \mathrm{E}(\xi^2) - (\mathrm{E}(\xi))^2 = 3(\Delta t)^2 - (\Delta t)^2 = 2(\Delta t)^2 \approx 0 \end{aligned}$$

- Similarly, if we define $\zeta = (\Delta t)(\Delta W_t)$, we have that:

$$\begin{aligned} \mathrm{E}(\zeta) &= 0 \\ \mathrm{V}(\zeta) &= \mathrm{E}(\zeta^2) - (\mathrm{E}(\zeta))^2 = (\Delta t)^2 \, \mathrm{E}(\xi) = (\Delta t)^3 \approx 0 \end{aligned}$$

- Hence, $(\Delta W_t)^2 \approx \Delta t$ and $(\Delta t)(\Delta W_t) \approx 0$ for small Δt.

Intuitive Ito's Lemma

- Consider a GBM process $\{S_t\}$ and a smooth function $f(\cdot)$.
- A second order Taylor approximation around S_t implies:

$$f(S_t + \Delta S_t) \approx f(S_t) + f'(S_t)(\Delta S_t) + \frac{1}{2} f''(S_t)(\Delta S_t)^2$$

- Using the results derived before:

$$\begin{aligned}
(\Delta S_t)^2 &= (\mu S_t \Delta t + \sigma S_t \Delta W_t)^2 \\
&= (\mu S_t)^2 \underbrace{(\Delta t)^2}_{\approx 0} + 2\mu\sigma (S_t)^2 \underbrace{(\Delta t)(\Delta W_t)}_{\approx 0} + (\sigma S_t)^2 \underbrace{(\Delta W_t)^2}_{\approx \Delta t} \\
&\approx \sigma^2 S_t^2 \Delta t
\end{aligned}$$

- We can finally conclude that:

$$\Delta f(S_t) \approx \left(\mu S_t f'(S_t) + \frac{1}{2} \sigma^2 S_t^2 f''(S_t) \right) \Delta t + \sigma S_t f'(S_t) \Delta W_t$$

Ito's Lemma

- The continuous-time analog of the previous analysis is as follows.
- As before, we consider a GBM process $\{S_t\}$ given by:

$$dS = \mu S dt + \sigma S dW$$

  and a smooth function $F(\cdot)$.
- Define a new process $\{X_t\}$ as $X_t = F(S_t)$ for all $t \in [0, T]$.
- Ito's lemma states that:

$$dF = \left(\mu S F'(S) + \frac{1}{2}\sigma^2 S^2 F''(S) \right) dt + \sigma S F'(S) dW$$

Ito Calculus Rules

- It is usually more convenient to use the following results when working with stochastic processes defined through Brownian motions:

$$
\begin{aligned}
(dt)^2 &= 0 \\
(dt)(dW) = (dW)(dt) &= 0 \\
(dW)^2 &= dt
\end{aligned}
$$

- Ito's Lemma can then be restated as:

$$dF = F'(S)dS + \frac{1}{2}F''(S)(dS)^2$$

where

$$(dS)^2 = (\mu S dt + \sigma S dW)^2 = \sigma^2 S^2 dt$$

Solving for GBM

- Define $X = \ln(S)$, which implies $S = e^X$.
- We have that $F'(S) = 1/S$ and $F''(S) = -1/S^2$, which implies that:

$$dX = \left(\mu - \frac{1}{2}\sigma^2\right) dt + \sigma dW$$

- We can then solve for X_T:

$$\begin{aligned} X_T - X_0 = \int_0^T dX &= \int_0^T \left(\mu - \frac{1}{2}\sigma^2\right) dt + \int_0^T \sigma dW \\ &= \left(\mu - \frac{1}{2}\sigma^2\right) T + \sigma W_T \end{aligned}$$

- We can finally conclude that:

$$S_T = S_0 \exp\left(\left(\mu - \frac{1}{2}\sigma^2\right) T + \sigma W_T\right)$$

Properties of Stock Prices Following a GBM

- The previous result can be rewritten as:

$$\ln(S_T) = \ln(S_0) + \left(\mu - \frac{1}{2}\sigma^2\right) T + \sigma W_T$$

- We can conclude that $\ln(S_T) \sim N(m, s^2)$, where:

$$m = \ln(S_0) + \left(\mu - \frac{1}{2}\sigma^2\right) T$$
$$s = \sigma\sqrt{T}$$

- In other words, S_T is lognormally distributed with mean m and variance s^2.

Calculating a Confidence Interval on the Stock Price

Example 1

Consider a stock whose price at time t is given by S_t and that follows a GBM. The expected return is 12% per year and the volatility is 25% per year. The current spot price is \$25. If we denote $X_T = \ln(S_T)$ and take $T = 0.5$, we have that:

$$\begin{aligned} \mathrm{E}(X_T) &= \ln(25) + \left(0.12 - 0.5(0.25)^2\right)(0.5) = 3.2633 \\ \mathrm{SD}(X_T) &= 0.25\sqrt{0.5} = 0.1768 \end{aligned}$$

Hence, the 95% confidence interval for S_T is given by:

$$[e^{3.2633-1.96(0.1768)}, e^{3.2633+1.96(0.1768)}] = [18.48, 36.96]$$

Therefore, there is a 95% probability that the stock price in 6 months will lie between \$18.48 and \$36.96.

Calculating the Moments of the Stock Price

- Some algebra reveals the expectation and standard deviation of S_T:

$$\mathrm{E}(S_T) = S_0 e^{\mu T}$$

$$\mathrm{SD}(S_T) = E(S_T)\sqrt{e^{\sigma^2 T} - 1}$$

Example 2

Consider a stock whose price at time t is given by S_t and that follows a GBM. The expected return is 12% per year and the volatility is 25% per year. The current spot price is \$25. The expected price and standard deviation 6 months from now are:

$$\mathrm{E}(S_T) = 25e^{0.12(0.5)} = \$26.55$$

$$\mathrm{SD}(S_T) = 26.55\sqrt{e^{0.25^2(0.5)} - 1} = \$4.73$$

Computing Partial Expectations

- Since $\ln(S_T) \sim \mathrm{N}(m, s^2)$. Then we have that:

$$\begin{aligned} \mathrm{E}\left(S_T \mathbb{1}_{\{S_T > K\}}\right) &= e^{m+\frac{1}{2}s^2}\,\Phi\left(\frac{m+s^2-\ln(K)}{s}\right) \\ &= S_0 e^{\mu T}\,\Phi\left(\frac{\ln(S_0/K)+(\mu+\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right) \\ \mathrm{E}\left(K \mathbb{1}_{\{S_T > K\}}\right) &= K\,\Phi\left(\frac{m-\ln(K)}{s}\right) \\ &= K\,\Phi\left(\frac{\ln(S_0/K)+(\mu-\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right) \end{aligned}$$

- It turns out that these results are everything we need in order to derive the Black-Scholes pricing formulas!

A Generalized Form of Ito's Lemma

- Most derivatives not only depend on the underlying asset but also depend on time since they have fixed expiration dates.
- The analysis we did before for Ito's Lemma generalizes easily to handle this case.
- Consider a non-dividend paying stock that follows a GBM:

$$dS = \mu S dt + \sigma S dW$$

  and a smooth function $F(S, t)$.
- Ito's Lemma in this case applies in the following form:

$$dF = \frac{\partial F}{\partial S} dS + \frac{1}{2} \frac{\partial^2 F}{\partial S^2} (dS)^2 + \frac{\partial F}{\partial t} dt$$

  where $(dS)^2 = \sigma^2 S^2 dt$.