Binomial Option Pricing

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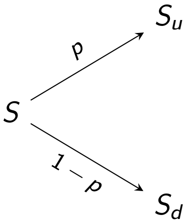
Binomial Trees

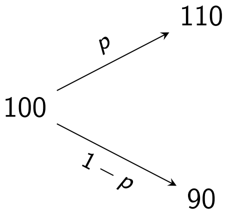
- One of the easiest ways to describe the evolution over time of a stock price is to use what in finance we call a binomial tree.
- At each point there are only two possibilities for the future stock price happening with probability p and $1-p$, respectively.



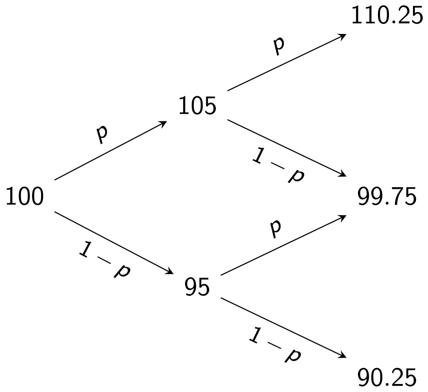
- It is common in finance to define $S_u = S \times u$ and $S_d = S \times d$, where u and d are the gross percentage increase and decrease of the stock price over the next period, respectively.

Example: One-Period Binomial Tree

- The current stock price is \$100.
- Next period, the asset can go up or down by 10% with probability p and $1 - p$, respectively.
- In this example $u = 1.10$ and $d = 0.90$.

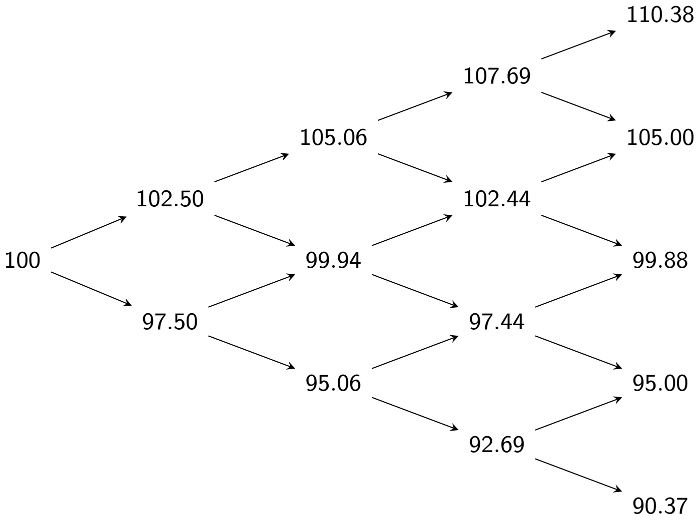


Example: Two-Period Binomial Tree



Note: Each period the asset can go up or down by 5%.

Example: Four-Period Binomial Tree



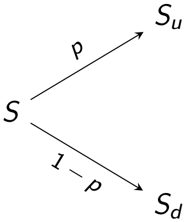
Note: Each period the asset can go up or down by 2.5%.

Recombinant Trees

- In the previous examples, going first *up* and then *down* is the same as going first *down* and then *up*.
- When this happens we say that the tree recombines.
  - Recombinant trees are very useful in modeling the stochastic behavior of financial assets because the number of nodes increases linearly with the number of periods, i.e after n periods there are $n+1$ possible nodes.
  - If the tree does not recombine then the number of nodes increases exponentially, i.e. after n periods there are 2^{n+1} possible nodes.
- In the following all trees we will work with will be recombinant.

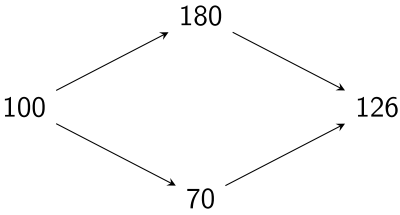
Recombinant Trees (cont'd)

- Consider the following node in a binomial tree:



- A recombinant tree is obtained whenever u and d are kept constant in each node of the tree.
- Note that in a recombinant tree u need not be equal to d.

- Is it the same for an asset to go up by 80% and then down by 30%, compared to first go down by 30% and then go up by 80%?
- Consider an asset whose current price is \$100.



- The tree recombines because of the associative property of multiplication:

$$S_{ud} = 100 \times 1.80 \times 0.70 = 100 \times 0.70 \times 1.80 = S_{du}$$

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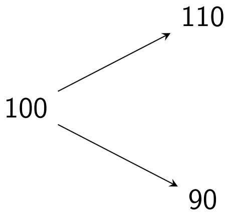
Pricing Options

- Consider a non-dividend paying stock that currently trades for \$100.
- Over the next 6-months the stock can go up or down by 10%.



Stock

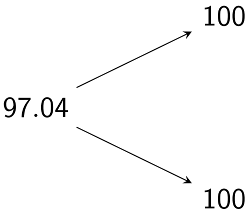
- The interest rate is 6% per year with continuous compounding.
- What should be the price of a European put option with maturity 6 months and strike price \$100?

The Binomial Tree of a Bond

- Consider a risk-free bond with maturity 6 months and face value equal to the strike price of the put, i.e., \$100.
- The price of the bond is:

$$B = 100e^{-0.06\times 6/12} = \$97.04$$

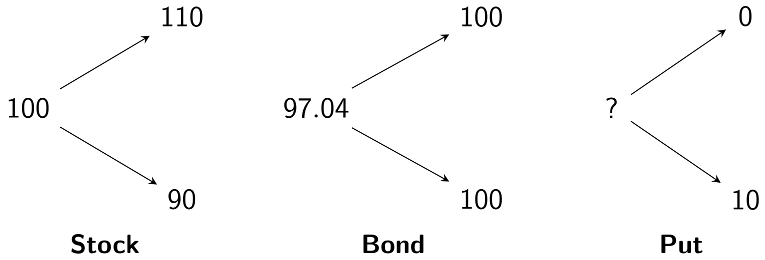
- The binomial tree for the bond is:



Bond

The Replicating Portfolio Approach

- We will price the option by using the stock and bonds to replicate the payoffs of the put.



- Say we purchase N_S units of the stock and N_B units of the bond.
- Such a portfolio would pay

$$\text{Payoff} = \begin{cases} 110 N_S + 100 N_B & \text{if } S = 110 \\ 90 N_S + 100 N_B & \text{if } S = 90 \end{cases}$$

Step 1: Replicating the Portfolio

- Furthermore, say we choose N_S and N_B such that payoff of the portfolio matches the payoff of the put, i.e.

$$\begin{aligned} 110N_S + 100N_B &= 0 \\ 90N_S + 100N_B &= 10 \end{aligned}$$

- We can solve for N_S and N_B to find:

$$\begin{aligned} N_S &= \frac{0 - 10}{110 - 90} = -0.50 \\ N_B &= -\frac{110}{100}N_S = (-1.1)(-0.5) = 0.55 \end{aligned}$$

Step 2: Pricing The Put

- Therefore, by shorting 0.50 units of the stock and going long 0.55 units of the bond we can exactly match the payoffs of the put.
- The price of the put must then match the price of the portfolio, otherwise there would be an arbitrage opportunity:

$$P = -0.5 \times 100 + 0.55 \times 97.04 = \$3.38$$

Example: Put Arbitrage

- What would happen in the previous analysis if the put was trading for \$3?

Long Put	-3.00
Short Synthetic Put	3.38
Total	0.38

$S_T = 110$

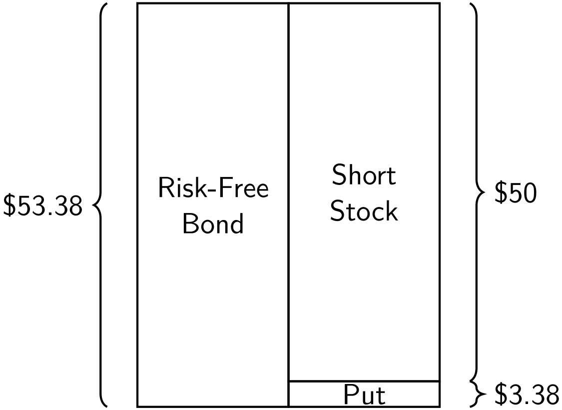
Long Put	0
Short Synthetic Put	0
Total	0

$S_T = 90$

Long Put	10
Short Synthetic Put	-10
Total	0

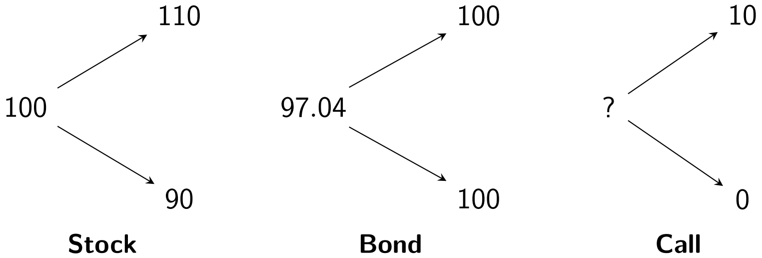
Put Leverage

- The replication analysis shows that the put can be seen as an investment in the risk-free bond that is financed in part by shorting stocks.



Replicating A Call Option

- Consider now a European call option with the same maturity and strike price as the put.



- As before, we replicate the payoffs of the call option by trading the stock and the bond:

$$110N_S + 100N_B = 10$$
$$90N_S + 100N_B = 0$$

Pricing the Call

- We can solve for N_S and N_B to find:

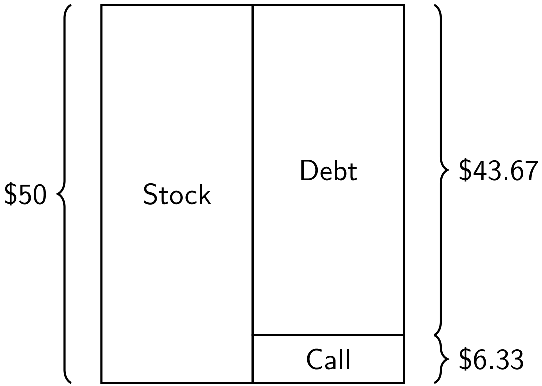
$$N_S = \frac{10 - 0}{110 - 90} = 0.50$$
$$N_B = -\frac{90}{100} N_S = (-0.90)(0.5) = -0.45$$

- Therefore, by buying 0.50 units of the stock and shorting 0.45 units of the bond we can exactly match the payoffs of the call.
- The price of the call must then match the price of the portfolio, otherwise there would be an arbitrage opportunity:

$$C = 0.5 \times 100 - 0.45 \times 97.04 = \$6.33$$

Call Leverage

- The replication analysis reveals that the call can be seen as a levered position on the stock.



Replicating A Generic Derivative

- The analysis so far suggests that we can generalize the replicating approach to price any derivative.

$$
\begin{tikzcd}
 & S_u & & & F & & & X_u \\
S \arrow[ur] \arrow[dr] & & & B \arrow[ur] \arrow[dr] & & & X \arrow[ur] \arrow[dr] & \\
 & S_d & & & F & & & X_d
\end{tikzcd}
$$

Stock **Bond** **Derivative**

- As before, we start by replicating the payoffs of the derivative by trading the stock and the bond:

$$
\begin{aligned}
S_u N_S + F N_B &= X_u \\
S_d N_S + F N_B &= X_d
\end{aligned}
$$

Pricing the Derivative

- We can solve for N_S and N_B to find:

$$N_S = \frac{X_u - X_d}{S^u - S^d}$$
$$N_B = \frac{X_u - S_u N_S}{F} = \frac{X^d - S^d N_S}{F}$$

- The price of the derivative must then match the price of the portfolio, otherwise there would be an arbitrage opportunity:

$$X = N_S S + N_B B$$

- The number of shares N_S needed to replicate the derivative is called the delta of the instrument.

Summary

Pricing a Derivative in the Binomial Model

In a one-period binomial model, the price X of a European call or put option with strike K and maturity T takes the form:

$$X = N_S S + N_B B$$

where

- S is the current stock price
- B is the price of a risk-free zero-coupon bond with face value F and maturity T
- N_S and N_B are the number of shares and risk-free bonds, respectively, needed to replicate the derivative

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The Risk-Neutral Approach

- In replicating the payoffs of the option, we never used the actual probabilities.
- As a matter of fact, these probabilities might even change based on whose thinking about the asset.
- Since the previous reasoning is silent about the probabilities and the type of investor pricing the asset, we can assume in our reasoning that all investors are risk neutral.
- Even if this is not true in real markets, such assumption would not affect **the logic** of the replicating-portfolio argument.

The Real Probabilities Are Irrelevant

- In a world populated by risk-neutral investors, all expected payoffs should be discounted at the risk-free rate, regardless of their riskiness.
- Hence, the price of the stock in this world, which is \$100, should be equal to the expected payoff discounted at the risk-free rate:

$$100 = (110p + 90(1-p))e^{-0.06\times 6/12}$$

- We can reverse-engineer the probability of the stock going up that makes consistent valuations in this world:

$$p = \frac{100e^{0.06\times 6/12} - 90}{110 - 90} = 0.6522$$

Pricing the Call and Put Again

- The price of the call is also equal to the expected payoff under this risk-neutral probability, discounted at the risk-free rate:

$$C = (10p + 0(1-p))e^{-0.06\times 6/12} = \$6.33$$

- Similarly, for the put we have that:

$$P = (0p + 10(1-p))e^{-0.06\times 6/12} = \$3.38$$

- Of course, the prices are the same as before since both approaches are consistent with each other.

Example 1

- A non-dividend paying stock trades at \$50 and over the next 6-months can go up to \$60 or down \$40.
- The risk-free rate is 6% per year with continuous compounding.
- Compute the price of a European call option expiring in 6 months with strike price \$48.

Example 1: Solution

- The risk-neutral probability of the stock moving up is:

$$p = \frac{50e^{0.06\times 6/12} - 40}{60 - 40}$$

- The price of the call is:

$$C = (12p + 0(1 - p))e^{-0.06\times 6/12} = 6.71$$

Example 2

- A non-dividend paying stock trades at \$120 and over the next 3-months can increase or decrease by 10%.
- The risk-free rate is 5% per year with continuous compounding.
- Compute the price of an asset that pays in 3 months \$100 if the stock increases in price and \$200 otherwise.

Example 2: Solution

- The stock can move up to 132 or down to 108.
- The risk-neutral probability of the stock moving up is:

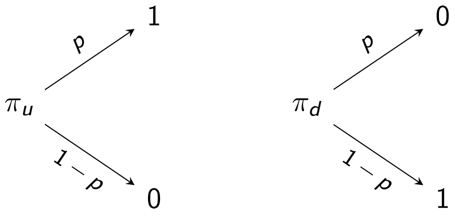
$$p = \frac{120e^{0.05 \times 3/12} - 108}{132 - 108}$$

- The price of the asset is:

$$X = (100p + 200(1 - p))e^{-0.05 \times 3/12} = 141.93$$

State Prices

- The risk-neutral probabilities are intimately related to the so-called Arrow-Debreu securities depicted below.



- The price of each security is then the expected payoff using the risk-neutral probabilities, discounted at the risk-free rate:

$$\pi_u = (1p + 0(1-p))e^{-rT}$$
$$\pi_d = (0p + 1(1-p))e^{-rT}$$

Example 3

- A non-dividend paying stock trades at \$120 and over the next 3-months can increase or decrease by 10%.
- The risk-free rate is 5% per year with continuous compounding.
    1. Compute the price of an asset that pays in 3 months \$1 if the stock price increases and \$0 otherwise.
    2. Compute the price of an asset that pays in 3 months \$0 if the stock price increases and \$1 otherwise.
    3. Using the previous results, compute the price of an asset that pays in 3 months \$100 if the stock increases in price and \$200 otherwise.

Example 3: Solution

- Using the same risk-neutral probabilities as in Example 2, we have that:
  1. The price of an asset that pays in 3 months \$1 if the stock price increases and \$0 otherwise is:

$$\pi_u = (1p + 0(1-p))e^{-0.05\times 3/12} = 0.5559$$

  2. The price of an asset that pays in 3 months \$0 if the stock price increases and \$1 otherwise is:

$$\pi_d = (0p + 1(1-p))e^{-0.05\times 3/12} = 0.4317$$

  3. the price of an asset that pays in 3 months \$100 if the stock increases in price and \$200 otherwise is:

$$X = 100\pi_u + 200\pi_d = 141.93$$

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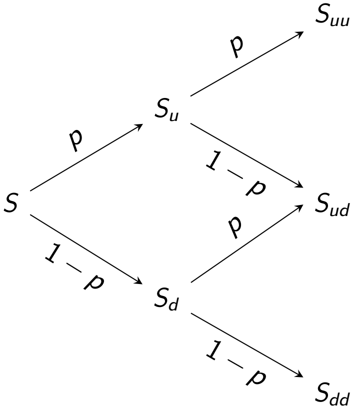
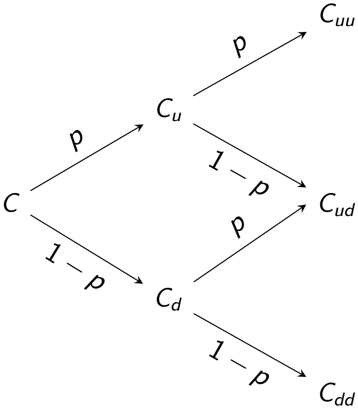
The Two Period Binomial Model

- We now extend the economy to two periods
- The spot rate is given by S
- Each period the spot rate goes up by u or goes down by d with risk-neutral probabilities p and $1 - p$, respectively
- Therefore:
  - $S_u = S \times u$ and $S_d = S \times d$
  - $S_{uu} = S_u \times u$, $S_{ud} = S_u \times d = S_d \times u = S_{du}$ and $S_{dd} = S_d \times d$
- A call option expiring at T and strike K trades at C
- The time-step is then $\Delta T = T/2$

Two Period Tree for the Spot and Call Option



Stock



Call

Pricing a Call Option

- The call price at expiration is the intrinsic value of the option:

  $C_{uu} = \max(S_{uu} - K, 0)$, $C_{ud} = \max(S_{ud} - K, 0)$, and $C_{dd} = \max(S_{dd} - K, 0)$

- If the spot rate goes up during the next 6 months, the call price must be equal to:

$$C_u = (pC_{uu} + (1-p)C_{ud})\, e^{-r\Delta t}$$

- If the spot rate goes down we have that:

$$C_d = (pC_{du} + (1-p)C_{dd})\, e^{-r\Delta t}$$

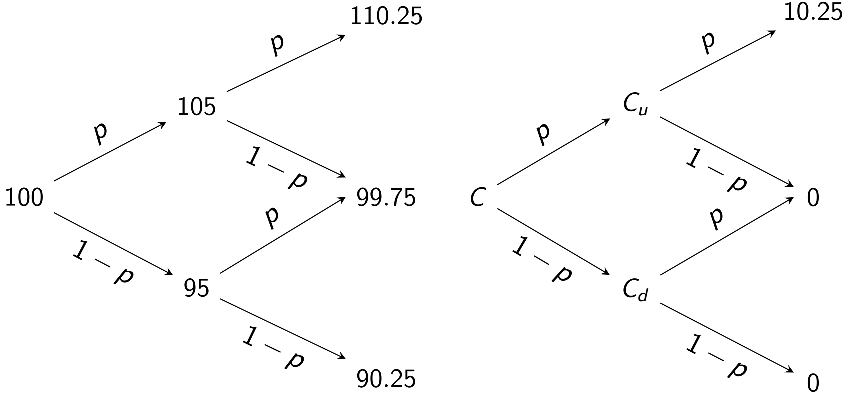
- Finally, the current value of the option must be:

$$C = (pC_u + (1-p)C_d)\, e^{-r\Delta t}$$

Example 4

- Let us price a European call option written on a non-dividend paying stock using a two-step binomial model.
- The current stock price is \$100, and it can go up or down by 5% each period for two periods.
- Each period represents 3-months, i.e. $\Delta t = 0.25$.
- The risk-free rate is 6% per year (continuously compounded).
- Compute the price of a European call option with maturity 6 months and strike \$100.

Two Period Tree for the Spot and Call Option



Stock **Call**

Computing the Risk-Neutral Probabilities

- The risk-neutral probability of an up-move is then:

$$p = \frac{e^{r\Delta t} - d}{u - d} = \frac{e^{0.06\times 0.25} - 0.95}{1.05 - 0.95} = 0.6511$$

- The risk-neutral probability of a down-move is just $1 - p = 0.3489$

Pricing the European Call Option

- We then compute the price of the call in 3-months if the stock price moves up:

$$C_u = (10.25 \times p + 0 \times (1 - p))\, e^{-0.06 \times 0.25} = \$6.57$$

- Next, we compute the price of the call in 3-months if the stock price moves down:

$$C_d = (0 \times p + 0 \times (1 - p))\, e^{-0.06 \times 0.25} = \$0$$

- Finally, we compute the price of the call:

$$C = (6.57 \times p + 0 \times (1 - p))\, e^{-0.06 \times 0.25} = \$4.22$$

Pricing a European Put Option

- We can use the risk-neutral probabilities to price a European put with the same characteristics.
- We compute the price of the put in 3-months if the stock price moves up:

$$P_u = (0 \times p + 0.25 \times (1 - p))\, e^{-0.06 \times 0.25} = \$0.08$$

- Next, we compute the price of the put in 3-months if the stock price moves down:

$$P_d = (0.25 \times p + 9.75 \times (1 - p))\, e^{-0.06 \times 0.25} = \$3.51$$

- Finally, we compute the price of the put:

$$P = (0.08 \times p + 3.51 \times (1 - p))\, e^{-0.06 \times 0.25} = \$1.26$$

Making the Tree Consistent with Observed Volatility

- It is possible to relate the up and down movements to the risk-neutral volatility observed in the market.
- It can be shown that over an interval Δt, the choice $u = e^{\sigma\sqrt{\Delta t}}$ and $d = 1/u$ produce a binomial model consistent with the Black-Scholes model of a Geometric Brownian Motion (GBM).
- Note that in this case $u \times d = 1$, so the tree on average does not have a drift.
- The risk-neutral drift, however, is incorporated into the risk-neutral probabilities.