Futures and Forward Prices

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The Forward or Futures Price

- The futures or forward price of of a non-dividend paying asset with maturity T years is given by:

$$F = Se^{rT}$$

  where S denotes the spot price of the asset and r is the risk-free rate expressed per year with continuous compounding.

Example: Forward Price of a Non-Dividend Paying Stock

- Consider a non-dividend paying stock trading at $40.
- The risk-free rate is 5% per year with continuous compounding.
- What is the 3-month forward price?

$$F = 40e^{0.05\times 3/12} = \$40.50$$

- What if the forward price was higher or lower than $40.50?

Example: Forward Price Arbitrage (1)

- Suppose that the spot price of a non-dividend-paying stock is \$40, the 3-month forward price is \$43 and the 3-month US\$ interest rate is 5% per year with continuous compounding.
- Is there an arbitrage opportunity?

	$T = 0$	$T = 3/12$
Short forward	0.00	$43 - S_T$
Borrow	42.47	-43
Long stock	-40.00	S_T
Total	2.47	0

- Yes, the forward price is too high!

Example: Forward Price Arbitrage (2)

- Suppose that the spot price of non-dividend paying stock is \$40, the 3-month forward price is US\$39 and the 3-month US\$ interest rate is 5% per year with continuous compounding.
- Is there an arbitrage opportunity?

	$T = 0$	$T = 3/12$
Long forward	0.00	$S_T - 39$
Invest	-38.52	39
Short stock	40.00	$-S_T$
Total	1.48	0

- Yes, the forward price is too low!

Example: Gold Futures Price

- Suppose that gold spot is currently \$1,870.60, and consider a futures contract on gold expiring in one year.
- Assume that the cost of storing gold is negligible and there are no additional benefits accruing from owning gold.
- The risk-free rate is 5% per year with continuous compounding.
- Then, the no-arbitrage futures price of gold is:

$$F = 1870.60e^{0.05} = \$1{,}966.51.$$

Valuing an Existing Forward Contract

- A forward contract is worth zero when it is first negotiated.
- Afterwards it may have a positive or negative value.
- Suppose that K is the delivery price and F is the forward price for a contract that would be negotiated today.
- By considering the difference between a contract with delivery price K and a contract with delivery price F we can deduce that:
  - The value of a long forward contract is $(F - K)e^{-rT}$.
  - The value of a short forward contract is $(K - F)e^{-rT}$.

Example: Valuing an Existing Forward Position

- You entered into a short forward contract some time ago on a non-dividend paying asset when the forward price was \$200.
- Today the contract has 6 months until maturity and the current forward price is \$190.
- The current risk-free rate is 5% per year with continuous compounding.
- To compute the current value of the short forward position, we could imagine what would happen if we buy a forward today.
  - That would lock-in a certain cash flow in six months of $200 - 190 = \$10$.
  - The present value today of this cash flow is $10e^{-0.05\times 6/12} = \$9.75$ which is the value of the short forward contract.

Forward vs Futures Prices

- If interest rates are constant, forward and futures prices are the same.
- When interest rates are uncertain, futures and forwards are in theory not exactly the same.
- A strong positive correlation between interest rates and the asset price implies the futures price is higher than the forward price as would be the case for Eurodollar futures.
- A strong negative correlation implies the reverse.

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Assets Paying Cash Dividends

- The futures or forward price of an asset paying cash dividends is given by:
$$F = (S - D)e^{rT}$$
where D is the present value of the dividends or income earned during life of forward contract.
- Note that D could be negative if the asset requires to pay for storage and does not provide any other source of income.

Example: Forward Price of a Dividend Paying Stock

- Consider a stock that currently trades at \$50.
- The stock is expected to pay dividends of \$1.15 and \$1.20 in two and five months, respectively.
- The risk-free rate is 5% per year with continuous compounding.
- The present value of the dividends paid during the life of the forward contract is:

$$D = 1.15e^{-0.05\times 2/12} + 1.20e^{-0.05\times 5/12} = 2.32$$

- The 6-month forward price of the stock is:

$$F = (50 - 2.32)e^{0.05\times 6/12} = 48.89$$

Assets Paying a Dividend Yield

- The futures price of a dividend-yield paying asset is given by:

$$F = Se^{(r-q)T}$$

where S is the spot price of the asset, T is the maturity of the futures contract, q is the continuous dividend or convenience yield, and r denotes the continuously compounded interest rate.

Index Futures

- A stock index can be viewed as an investment asset paying a dividend yield q.
- The futures-spot price relationship is

$$F = Se^{(r-q)T}$$

  where q is the dividend yield of the portfolio tracking the index.
- In this relationship, S closely tracks the level of the index as long as it is possible to trade its constituents.

Example: Index Futures

- Consider an index tracking a portfolio of stocks that pays a dividend yield of 3% per year with continuous compounding.
- The index is currently at 4,300. The risk-free rate for all maturities is 1% per year continuously-compounded.
- What should be the 6-month futures price of the index?
- If we denote by F the futures price, then we have that:

$$F = 4300e^{(0.01-0.03)\times 6/12} = 4257.21$$

- Note that because the dividend yield is higher than the risk-free rate, the futures price is less than the current spot price.

Index Arbitrage

- Index arbitrage involves simultaneous trades in futures and many different stocks.
- Very often a computer is used to generate the trades.
- When $F > Se^{(r-q)T}$ an arbitrageur buys the stocks underlying the index and sells futures.
- When $F < Se^{(r-q)T}$ an arbitrageur buys futures and shorts or sells the stocks underlying the index.
- Occasionally simultaneous trades are not possible and the theoretical no-arbitrage relationship between F and S does not hold.

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Currencies and Exchange Rates

- The exchange rate between two currencies is usually defined as the number of domestic currency units per unit of foreign currency.
- Note that you could always define it the other way around (indirect-quotes), but that could lead to mistakes.
- Consider the EUR/USD exchange rate:
  - The **quote** currency is the US dollar (USD)
  - The **base** currency is the Euro (EUR)
- If the EUR/USD exchange rate is \$1.47/€
  - For a US investor, 1 Euro is worth \$1.47
  - In Europe, how many Euros is worth \$1?

$$\$1 = \frac{1}{1.47} = €0.68.$$

Direct Quotes for Exchange Rates

- Remember the street market convention:
  - A direct quote is the price of 1 unit of base currency expressed in the quote currency
  - For example, the direct quote of the EUR/USD could be $S = \$1.4380/€$ and represents the price in USD of 1 EUR.
- The market convention of calling this exchange rate EUR/USD might be misleading since it represents the number of USD per EUR, i.e. $\$1.4380 \Leftrightarrow €1$.
- Some currency pairs such as EUR/USD or GBP/USD use the USD as the quote currency.
- However, most currency pairs are expressed using the dollar as the base currency, i.e., USD/JPY, USD/CNY, USD/CLP, etc.

Currency Forward

- A foreign currency is analogous to a security providing a yield.
- The yield is the foreign risk-free interest rate.
- It follows that if r^* is the foreign risk-free interest rate

$$F = Se^{(r-r^*)T}$$

Example: Currency Forward

- The current GBP/USD exchange rate is 1.30.
- The interest rates in USD and GBP are 1% and 3% per year with continuous compounding, respectively.
- The 9-month GBP/USD forward price is then

$$F = 1.30e^{(0.01-0.03)\times 9/12} = 1.2806,$$

or

$$10{,}000 \times (1.2806 - 1.3000) = -193.5 \text{ forward-points.}$$

Commodity Futures

- For commodities, the convenience yield y represents the net dividend paid by the commodity and the futures price is computed as:

$$F = S_0 e^{(r-y)T}$$

- The cost of carry, c, is the storage cost plus the interest costs less the income earned, so that for an investment or consumption asset we have that:

$$F = S_0 e^{cT}$$

Example: Oil Futures

- Suppose that the spot price of oil is \$95 per barrel.
- The 1-year US\$ interest rate is 5% per year with continuous compounding.
- The convenience yield is 2% per year.
- The 1-year oil futures price is

$$F = 95e^{0.05-0.02} = \$97.89.$$

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Futures Prices & Expected Future Spot Prices

- Suppose that μ is the expected return required by investors in an asset.
- We can invest Fe^{-rT} at the risk-free rate and enter into a long futures contract to create a cash inflow of S_T at maturity.
- This shows that

$$Fe^{-rT} = \mathrm{E}(S_T)e^{-\mu T}$$

or

$$F = \mathrm{E}(S_T)e^{(r-\mu)T}$$

No Systematic Risk	$\mu = r$	$F = \mathrm{E}(S_T)$
Positive Systematic Risk	$\mu > r$	$F < \mathrm{E}(S_T)$
Negative Systematic Risk	$\mu < r$	$F > \mathrm{E}(S_T)$