Options on Stock Indices and Currencies

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Outline

Pricing Formulas

Black-Scholes Model for an Asset Paying a Dividend Yield

Consider an asset S that pays a continuous yield q and that follows a GBM under the risk-neutral measure:

$$dS = (r - q)Sdt + \sigma SdW$$

The price of European call and put options with strike price K and time-to-maturity T are given by:

$$C = Se^{-qT}\,\Phi(d_1) - Ke^{-rT}\,\Phi(d_2)$$
$$P = Ke^{-rT}\,\Phi(-d_2) - Se^{-qT}\,\Phi(-d_1)$$

where

$$d_1 = \frac{\ln(S/K) + (r - q + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} \text{ and } d_2 = d_1 - \sigma\sqrt{T}$$

Modeling Dividends

- It is usually convenient to model dividends as a percentage yield paid over time.
- We will denote the continuously-compounded dividend yield by q.
- The asset S then pays every instant t a dividend of $qS_t\Delta t$.
- Therefore, if you purchase one unit of the asset at time t for S_t, the value of the portfolio at time $t + \Delta t$ will be $S_{t+\Delta t} + qS_t\Delta t$.
- In practice, this is the approach used to model options on stock indices and currencies, although some practitioners also use it to model individual stocks as well.

Replicating A Derivative

- Consider a derivative H with maturity T written on an asset S that pays a continuous dividend yield q.
- As we did before, the derivative can be replicated by buying (or selling) α units of the stock and β units of a bond with face value K and maturity T, respectively.
- If we call V the value of such replicating portfolio, we have that:

$$V_t = \begin{cases} \alpha_t S_t + \beta_t B_t & \text{if } t < T \\ H_T & \text{if } t = T \end{cases}$$

- At time $t + \Delta t$, the value of the replicating portfolio is:

$$V_{t+\Delta t} = \alpha_t (S_{t+\Delta t} + q S_t \Delta t) + \beta_t B_{t+\Delta t},$$

which implies that:

$$\Delta V_t = \alpha_t (\Delta S_t + q S_t \Delta t) + \beta_t \Delta B_t.$$

Replication in Continuous-Time

- As $\Delta t \to 0$, we have that:

$$
\begin{aligned}
dV &= \alpha(dS + qSdt) + \beta dB \\
&= \alpha(dS + qSdt) + r(\beta B)rdt \\
&= \alpha(dS + qSdt) + r(V - \alpha S)dt \\
&= (rV - (r-q)\alpha S)\, dt + \alpha dS
\end{aligned}
$$

- Also, Ito's Lemma implies that:

$$
dV = \left(\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t}\right) dt + \frac{\partial V}{\partial S} dS = (rV - (r-q)\alpha S)\, dt + \alpha dS
$$

The Risk-Neutral Process for the Underlying Asset

- Again, choosing $\alpha = \frac{\partial V}{\partial S}$ implies that:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - q)S\frac{\partial V}{\partial S} - rV = 0$$

  with boundary condition $V_T = H_T$.

- Using the same logic as before, we conclude that S follows a GBM under the risk-neutral measure given by:

$$dS = (r - q)Sdt + \sigma SdW$$

Pricing a Forward Contract

- We can use the risk-neutral approach to price a long forward contract with maturity T and forward price F.

$$
\begin{aligned}
V_t &= e^{-r(T-t)} \operatorname{E}_t(S_T - F) \\
&= e^{-r(T-t)}(S_t e^{(r-q)(T-t)} - F) \\
&= S_t e^{-q(T-t)} - F e^{-r(T-t)}
\end{aligned}
$$

- The forward price F is determined such that at inception the value of the contract is zero:

$$
V = Se^{-qT} - Fe^{-rT} = 0 \Rightarrow F = Se^{(r-q)T}
$$

- The value of the long position, in general, will change over time.

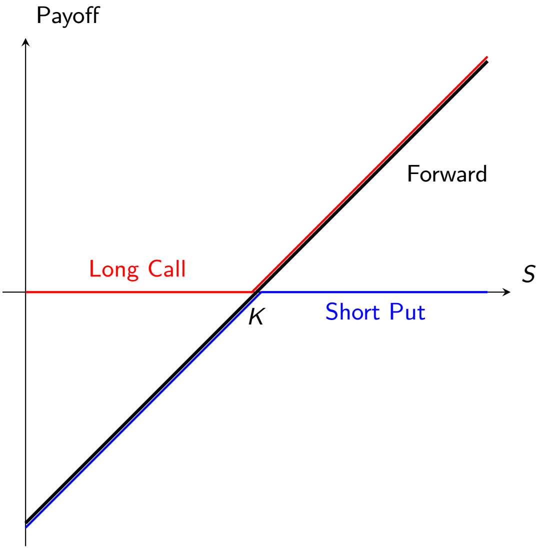
Put-Call Parity and Forward Contracts

- For European options with strike K and maturity T written on an asset that pays a dividend yield q, the following relationship known as put-call parity must hold:

$$C - P = Se^{-qT} - Ke^{-rT}$$

- The right hand-side of this expression is the cost of a forward contract with forward price K.
- The left hand-side says that a forward contract can be synthesized by buying a call and selling a put.

Payoff Diagram for Forward Contract



Pricing a European Call Option

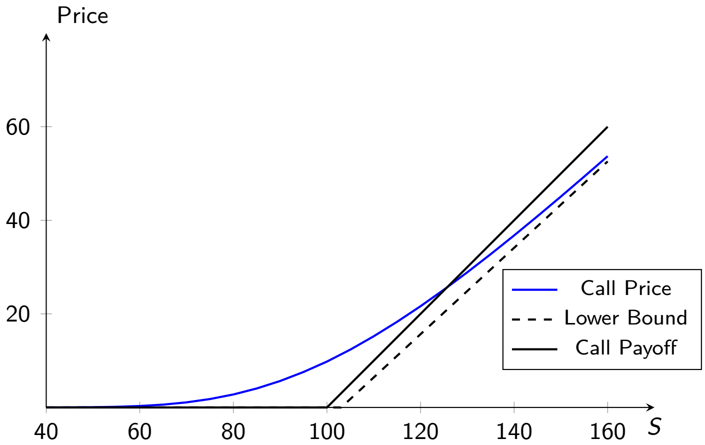
- As before, we can price a European call option written on the asset with maturity T and strike price K:

$$
\begin{aligned}
C &= e^{-rT}\,\mathrm{E}\left((S_T - K)\mathbb{1}_{\{S_T>K\}}\right) \\
&= e^{-rT}\,\mathrm{E}\left(S_T\mathbb{1}_{\{S_T>K\}}\right) - e^{-rT}\,\mathrm{E}\left(K\mathbb{1}_{\{S_T>K\}}\right) \\
&= Se^{-qT}\,\Phi(d_1) - Ke^{-rT}\,\Phi(d_2)
\end{aligned}
$$

where

$$
\begin{aligned}
d_1 &= \frac{\ln(S/K) + (r - q + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} \\
d_2 &= d_1 - \sigma\sqrt{T}
\end{aligned}
$$

Call Premium vs. Spot Price



The plot displays the Black-Scholes call premium $C(S)$ where $r = 0.05$, $q = 0.08$, $\sigma = 0.30$, $T = 1$ and $K = 100$. It also shows the call option payoff given by $\max(S - 100, 0)$ and the lower bound for a European call given by $\max(Se^{-0.08} - 100e^{-0.05}, 0)$.

Pricing a European Put Option

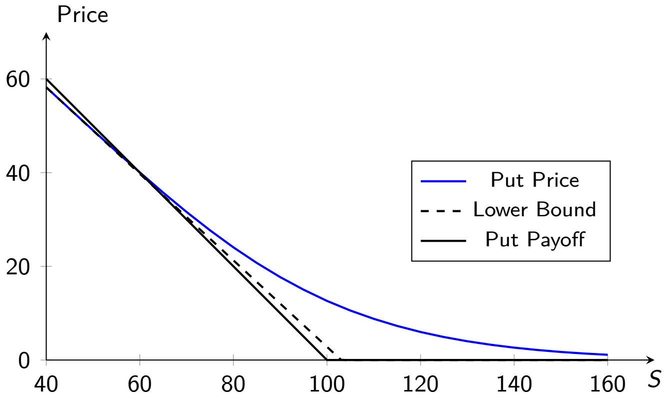
- Consider now a European put option with the same characteristics as the previous call.
- According to put-call parity, it must be the case that:

$$C - P = Se^{-qT} - Ke^{-rT}$$

- Hence,

$$\begin{aligned}
P &= C - (Se^{-qT} - Ke^{-rT}) \\
&= Se^{-qT}\Phi(d_1) - Ke^{-rT}\Phi(d_2) - (Se^{-qT} - Ke^{-rT}) \\
&= Ke^{-rT}(1 - \Phi(d_2)) - Se^{-qT}(1 - \Phi(d_1)) \\
&= Ke^{-rT}\Phi(-d_2) - Se^{-qT}\Phi(-d_1)
\end{aligned}$$

Put Premium vs. Spot Price



The plot displays the Black-Scholes put premium $P(S)$ where $r = 0.05$, $q = 0.08$, $\sigma = 0.30$, $T = 1$ and $K = 100$. It also shows the put option payoff given by $\max(100 - S, 0)$ and the lower bound for a European put given by $\max(100e^{-0.05} - Se^{-0.08}, 0)$.

Example 1

A stock that pays a continuous dividend yield of 8% currently trades for \$100. The instantaneous volatility of returns is 30% per year and the risk-free rate is 5% per year, continuously compounded and constant for all maturities. Consider ATM call and put options written on the stock with maturity 10 months. Then,

$$d_1 = \frac{\ln(100/100) + (0.05 - 0.08 + 0.5(0.30)^2)(10/12)}{0.30\sqrt{10/12}} = 0.0456$$

$$d_2 = 0.0456 - 0.30\sqrt{10/12} = -0.2282$$

Therefore, $\Phi(d_1) = 0.5182$ and $\Phi(d_2) = 0.4097$, which implies that:

$$C = 100e^{-0.08(10/12)}(0.5182) - 100e^{-0.05(10/12)}(0.4097) = \$9.18$$

$$P = 100e^{-0.05(10/12)}(1 - 0.4097) - 100e^{-0.08(10/12)}(1 - 0.5182) = \$11.54$$

Delta of European Call and Put Options

- For an asset that a pays a continuous dividend yield q, we have that for a European call option:

$$\frac{\partial C}{\partial S} = e^{-qT}\Phi(d_1)$$

- We can see that if $q > 0$, the number of shares required to hedge the call is lower than in the case of a non-dividend paying asset.
  - The shares that you buy to hedge the call grow over time at the rate q, which means that you need to buy less.
- Similarly, for a European put option we have that:

$$\frac{\partial P}{\partial S} = -e^{-qT}\Phi(-d_1)$$

Example 2

In the previous example, we found that $\Phi(d_1) = 0.5182$ and $\Phi(d_2) = 0.4097$. Hence,

$$\frac{\partial C}{\partial S} = e^{-0.08(10/12)}(0.5182) = 0.4848$$
$$\frac{\partial P}{\partial S} = -e^{-0.08(10/12)}(1 - 0.5182) = -0.4507$$

This means that an OTC dealer who sells a call option needs to buy 0.4848 units of the asset while borrowing

$$100e^{-0.05(10/12)}(0.4097) = \$39.30$$

at the risk-free rate. To hedge a put option, the dealer needs to short-sell 0.4507 units of the asset and invest

$$100e^{-0.05(10/12)}(1 - 0.4097) = \$56.62$$

in the money-market account.

Outline

1. General Framework

2. Options on Indices

3. Options on Currencies

Options on Stock Indices

- Most stock indices such as the S&P 500 (SPX) do not reinvest their dividends.
- Hence, to replicate an option written on the index we can use a portfolio of stocks that mimics the value of the index and that will pay a dividend yield over time.
- We will assume that the replicating portfolio exactly matches the composition of the index at any point in time so that S_t represents both the value of the index and of the tracking portfolio.

SPX Options

- One of the most liquid option contracts in the world.
- Characteristics:
  - European style exercise
  - Cash settled
  - Each contract is written on 100 times the value of the index
- There are also mini-SPX index options written over XSP which is an index 10 times smaller than SPX.
- More information can be found at `https://cdn.cboe.com/resources/spx/spx-fact-sheet.pdf`

Example 3

The SPX index is currently at 4,251, has a dividend yield of 1.33% per year and an instantaneous volatility of 17% per year. The risk-free rate is 3% per year, continuously compounded and constant for all maturities. Say we want to compute the price of an SPX call option contract with maturity 3 months and strike 4,300. Then,

$$d_1 = \frac{\ln(4251/4300) + (0.03 - 0.0133 + 0.5(0.17)^2)(3/12)}{0.17\sqrt{3/12}} = -0.0432$$

$$d_2 = -0.0432 - 0.17\sqrt{10/12} = -0.1282$$

Hence, $\Phi(d_1) = 0.4828$ and $\Phi(d_2) = 0.4490$, which implies that:

$$C = 4{,}251e^{-0.0133(3/12)}(0.4828) - 4{,}300e^{-0.03(3/12)}(0.4490) = \$129.193$$

Therefore, a standard SPX call option contract should cost \$12,919.30, whereas a mini-SPX call option contract should trade for \$1,291.93.

Outline

Exchange Rates

- The (nominal) exchange rate between two currencies is the number of domestic currency units per unit of foreign currency.
  - You could always define it the other way around (indirect-quotes)
- Consider the EUR/USD exchange rate:
  - The **quote** currency is the US dollar (USD)
  - The **base** currency is the Euro (EUR)
- If the EUR/USD exchange rate is \$1.47/€
  - For a US investor, 1 Euro is worth \$1.47
  - In Europe, how many Euros is worth \$1?

$$\$1 = \frac{1}{1.47} = €0.68/\$.$$

- The exchange rate is a **relative price**.

Direct Quotes for Exchange Rates

- Remember the street market convention:
  - A direct quote is the price of 1 unit of base currency expressed in the quote currency
  - For example, the direct quote of the EUR/USD could be $S = \$1.4380/€$ and represents the price in USD of 1 EUR.
- The market convention of calling this exchange rate EUR/USD might be misleading
  - It is written EUR/USD, EUR-USD or EURUSD but it really represents the number of USD per EUR, i.e. $\$1.4380 \Leftrightarrow €1$.
  - Be careful, though, in some textbooks you might find it the other way around.
- Some currency pairs such as EUR/USD or GBP/USD use the USD as the quote currency.
- However, most currency pairs are expressed using the dollar as the base currency, i.e., USD/JPY, USD/CNY, USD/CLP, etc.

The Risk-Neutral Process for a Currency

- When working with currencies, it is usually convenient to denote by r the risk-free rate of the quote currency and by r^* the risk-free rate of the base currency.
- The risk-neutral process for an exchange-rate S expressed with direct-quotes is then:

$$dS = (r - r^*)Sdt + \sigma SdW$$

Forward Contracts on Currencies

- Using the previous notation, the forward price with maturity T for the currency is:

$$F = Se^{(r-r^*)T}$$

Example 4

The EUR/USD currently trades at \$1.18663. The continuously compounded 9-month risk-free rates in USD and EUR are 1.5% and 0.5% per year, respectively. The 9-month EUR/USD forward rate is then:

$$F = 1.18663e^{(0.015-0.005)(9/12)} = \$1.19556$$

or $+89.3$ forward-points.

Options on Currencies

- Options on currencies reveal an interesting relationship between the underlying asset and the numeraire used to express the price of the asset.
- Consider an American investor analyzing a **call** option on the EUR/USD with maturity 1-year, strike price \$1.25 over a notional of €1 million.
- From the point of the view of a European investor, that option is really a **put** on the USD/EUR with same maturity, strike price €0.80 over a notional of \$1.25 million.
- Hence, it is convenient to be explicit about the currency being bought and the one being sold when specifying the contract, i.e., we will talk about a **EUR call/USD put** when describing the previous contract.

Option Pricing Formulas for Currencies

- It is common to express the Black-Scholes formulas for options on currencies as a function of the corresponding forward price:

$$C = Fe^{-rT}\,\Phi(d_1) - Ke^{-rT}\,\Phi(d_2)$$
$$P = Ke^{-rT}\,\Phi(-d_2) - Fe^{-rT}\,\Phi(-d_1)$$

where

$$F = Se^{(r-r^*)T}$$
$$d_1 = \frac{\ln(F/K) + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}$$
$$d_2 = d_1 - \sigma\sqrt{T}.$$

At-The-Money-Forward Options

- An option with a strike price equal to its corresponding forward price is called at-the-money-forward (ATMF).
- Remember put-call parity for currencies:

$$C - P = Se^{-r^* T} - Ke^{-rT}$$

- When $K = Se^{(r-r^*)T}$ we have that $C - P = 0$, i.e., when the strike price is equal to the forward price a call and a put with the same maturity are worth the same.