American Options

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American vs. European Option Premium

- The main difference of American options compared to their European counterparts is that they can be exercised early.
- Knowing when to optimally exercise an American call or put is a challenging problem.
- This is the main reason why it is harder to price American options compared to European options.
- In the following $\mathcal{C}$ or $\mathcal{P}$ denote an American call or put respectively, whereas their European counterparts are denoted by C or P.
- Because an American option has all the benefits of a European option, but in addition has the possibility of early exercise, it has to be the case that its premium is at least as high as the premium on a European option with the same characteristics, i.e. we must have that $\mathcal{C} \geq C$ and $\mathcal{P} \geq P$.

American Call Option on a Non-Dividend Paying Asset

- It is never optimal to exercise early an American call option when the asset pays no dividends as long as the risk-free rate is positive.
- To see why, notice that if $r > 0$ and $T > 0$, then we have that:

$$\mathcal{C} \geq C \geq S - Ke^{-rT} > S - K.$$

- If it were optimal to exercise early, the price of the American call would be equal to its intrinsic value, violating the strict inequality.
  - Intuitively, in this case the time-value of a European call is strictly positive.
  - Hence, you destroy value if you exercise immediately, even if the option is ITM!
- This implies that for non-dividend paying stocks the value of an American call option with maturity T and strike price K is the same as the value of a European call option with the same characteristics.

Early-Exercise of American Options

- When there are cash dividends, it might be optimal to exercise early an American call option just before the stock goes ex-dividend.
- For American puts the situation is similar, even when on assets with no-dividends.
- In summary, exercising early is all about opportunity costs. If there is no opportunity cost in waiting, then it is never optimal to exercise early.

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Put-Call Parity

- For American options written on non-dividend paying stocks, the put-call parity holds as two inequalities if $r > 0$:

$$S - K \leq \mathcal{C} - \mathcal{P} \leq S - Ke^{-rT}$$

- Interestingly, these inequalities are reversed if $r < 0$:

$$S - Ke^{-rT} \leq \mathcal{C} - \mathcal{P} \leq S - K$$

- Both inequalities imply that put-call parity holds in the same way as for European options if $r = 0$:

$$\mathcal{C} - \mathcal{P} = S - K$$

Example 1

On 2/8/17, Facebook (FB) closing price was \$134.20. As of that date, the stock does not pay dividends. Options on FB as on other stocks are American. Let us consider options on FB with maturity date 6/17/17. This implies that $T = 129/365 = 0.35$. If we use a continuously compounded interest rate of 1.5% per year, we can compute the no-arbitrage bounds on these options.

Strike	Call Price	Put Price	$S - K$	$\mathcal{C} - \mathcal{P}$	$S - Ke^{-rT}$
120	17.30	2.50	14.20	14.80	14.83
125	13.61	3.75	9.20	9.86	9.86
130	10.25	5.45	4.20	4.80	4.89
135	7.45	7.60	-0.80	-0.15	-0.09
140	5.13	10.35	-5.80	-5.22	-5.06
145	3.47	13.65	-10.80	-10.18	-10.03

As can be seen from the table, the difference between calls and puts is well within the bounds predicted by the theory for all strikes.

Option Bounds

- We always have that the American call is greater than its intrinsic value and less than the current spot price, i.e.,

$$\max(S - K, 0) \leq \mathcal{C} \leq S$$

- For American put options we have a similar result:

$$\max(K - S, 0) \leq \mathcal{P} \leq K$$

- Note that these bounds are not necessarily the tightest, depending on whether ot not it might be worthwhile to exercise the option early.

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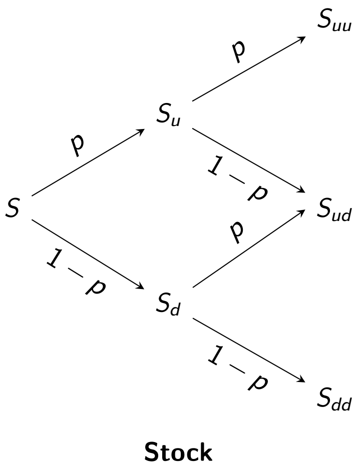
Early-Exercise and American Option Pricing

- In order to accommodate the binomial pricing framework to American options, we need to allow for the possibility of early exercise at any point in time.
- This means that we should always compare the intrinsic value of the option, i.e. the value of exercising now, with the value of continuing given by the discounted risk-neutral expected value of future payoffs.

A Two Period Binomial Model

- We will start by pricing an American put option on a non-dividend paying asset.
- To make the valuation problem interesting, we will use a two-period binomial tree.
- As with European options, the spot rate is given by S and each period the spot rate goes up by u or goes down by d with risk-neutral probabilities p and $1-p$, respectively
- Therefore:
  - $S_u = S \times u$ and $S_d = S \times d$
  - $S_{uu} = S_u \times u$, $S_{ud} = S_u \times d = S_d \times u = S_{du}$ and $S_{dd} = S_d \times d$
- An American put option expiring at T and strike K trades at $\mathcal{P}$
- The time-step is then $\Delta T = T/2$

Two Period Tree for the Spot and American Put Option



Pricing the American Put Option

- The American put price at expiration is just the intrinsic value of the option:

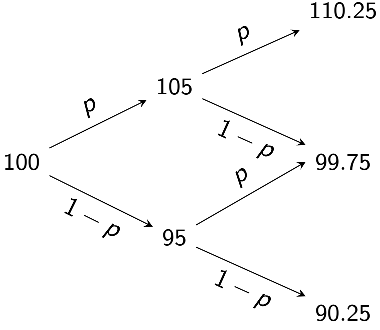
  $\mathcal{P}_{uu} = \max(K - S_{uu}, 0)$, $\mathcal{P}_{ud} = \max(K - S_{ud}, 0)$, and $\mathcal{P}_{dd} = \max(K - S_{dd}, 0)$

- If the spot rate goes up:
  - The intrinsic value of the put is $I_u = \max(K - S_u, 0)$.
  - The continuation value is $H_u = (p\mathcal{P}_{uu} + (1 - p)\mathcal{P}_{ud})e^{-r\Delta t}$.
  - If $I_u > H_u$ then the option should be exercised immediately, otherwise we should wait.
  - Therefore, the price of the option at that time is $\mathcal{P}_u = \max(H_u, I_u)$.
- Similarly, if the spot rate goes down, we have that $\mathcal{P}_d = \max(H_d, I_d)$.
- Finally, the value of the American put is given by $\mathcal{P} = \max(H, I)$, where:

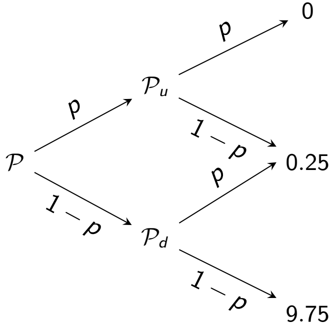
$$
\begin{aligned}
H &= (p\mathcal{P}_u + (1 - p)\mathcal{P}_d)\, e^{-r\Delta t} \\
I &= \max(K - S, 0)
\end{aligned}
$$

Example 2

Let us price an American put option with maturity 6 months and strike \$100 written on a non-dividend paying stock using a two-step binomial model. The current stock price is \$100, and it can go up or down by 5% each period for two periods. Each period represents 3-months, i.e. $\Delta t = 0.25$. The risk-free rate is 6% per year (continuously compounded).



Stock



Put

Example 2 (Cont'd)

The risk-neutral probability of an up-move is:

$$p = \frac{e^{r\Delta t} - d}{u - d} = \frac{e^{0.06\times 0.25} - 0.95}{1.05 - 0.95} = 0.6511$$

We then have that:

$$\begin{aligned} H_u &= (0 \times p + 0.25 \times (1-p))\, e^{-0.06\times 0.25} = 0.09 \\ I_u &= \max(100 - 105, 0) = 0 \end{aligned}$$

so that $\mathcal{P}_u = \max(0.09, 0) = 0.09$, and

$$\begin{aligned} H_d &= (0.25 \times p + 9.75 \times (1-p))\, e^{-0.06\times 0.25} = 3.51 \\ I_d &= \max(100 - 95, 0) = 5 \end{aligned}$$

which implies that $\mathcal{P}_d = \max(3.51, 5) = 5$.

Example 2 (Cont'd)

Finally,

$$
\begin{aligned}
H &= (0.09 \times p + 5 \times (1 - p))\, e^{-0.06 \times 0.25} = 1.78 \\
I &= \max(100 - 100, 0) = 0
\end{aligned}
$$

so that $\mathcal{P} = \max(1.78, 0) = \$1.78$. Note that the value of a European put with the same characteristics is $P = \$1.26$, implying that the early-exercise premium is $1.78 - 1.26 = \$0.52$. Also, you can verify that the value of an American call with the same characteristics is the same as the value of a Euroean call, which is \$4.22.

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Factors Affecting American Option Prices

Variable	America Call	American Put
Current stock price	$+$	$-$
Strike price	$-$	$+$
Time-to-expiration	$+$	$+$
Volatility	$+$	$+$
Risk-free rate	$+$	$-$